Approximation Algorithms for Restless Bandit Problems*

Sudipto Guha[†]

Kamesh Munagala[‡]

Peng Shi§

Abstract

The *restless bandit* problem is one of the most well-studied generalizations of the celebrated stochastic multi-armed bandit problem in decision theory. In its ultimate generality, the restless bandit problem is known to be PSPACE-Hard to approximate to any non-trivial factor, and little progress has been made on this problem despite its significance in modeling activity allocation under uncertainty.

We consider a special case that we call FEEDBACK MAB, where the reward obtained by playing each of n independent arms varies according to an underlying on/off Markov process whose exact state is only revealed when the arm is played. The goal is to design a policy for playing the arms in order to maximize the infinite horizon time average expected reward. This problem is also an instance of a Partially Observable Markov Decision Process (POMDP), and is widely studied in wireless scheduling and unmanned aerial vehicle (UAV) routing. Unlike the stochastic MAB problem, the FEEDBACK MAB problem does not admit to greedy index-based optimal policies. The state of the system at any time step encodes the beliefs about the states of different arms, and the policy decisions change these beliefs – this aspect complicates the design and analysis of simple algorithms.

We develop a novel and fairly general duality-based algorithmic technique that yields a surprisingly simple and intuitive $2 + \epsilon$ -approximate greedy policy to this problem. We then define a general sub-class of restless bandit problems that we term MONOTONE bandits, for which our policy is a 2-approximation. Our technique is robust enough to handle generalizations of these problems to incorporate various side-constraints such as blocking plays and switching costs. This technique is also of independent interest for other restless bandit problems, and we provide an example in non-preemptive machine replenishment. We finally show that our policies are closely related to the *Whittle index* that is widely used for its simplicity and efficiency of computation. In fact, not only is our policy just as efficient to compute as the Whittle index, but in addition, it provides surprisingly strong constant factor guarantees even in cases where the Whittle index is provably polynomially worse.

By presenting the first (and efficient) O(1) approximations for non-trivial instances of restless bandits as well as of POMDPs, our work initiates the study of approximation algorithms in both these contexts.

^{*}This paper combines and generalizes results presented in two papers [25] and [28] that appeared in the FOCS '07 and SODA '09 conferences respectively.

[†]Department of Computer and Information Sciences, University of Pennsylvania, Philadelphia PA 19104-6389. Email: sudipto@cis.upenn.edu. Research supported in part by an Alfred P. Sloan Research Fellowship, an NSF CAREER Award CCF-0644119

[‡]Department of Computer Science, Duke University, Durham NC 27708-0129. Email: kamesh@cs.duke.edu. Research supported by NSF via a CAREER award and grant CNS-0540347.

[§]Duke University, Durham NC 27708. Email: peng.shi@duke.edu. This research was supported by the Duke University Work-Study Program and by NSF award CNS-0540347.

1 Introduction

The celebrated multi-armed bandit problem (MAB) models the central trade-off in decision theory between exploration and exploitation, or in other words between learning about the state of a system and utilizing the system. In this problem, there are n competing options, referred to as "arms," yielding unknown rewards $\{r_i\}$. Playing an arm yields a reward drawn from an underlying distribution, and the information from the reward observed partially resolves its distribution. The goal is to sequentially play the arms in order to maximize reward obtained over some time horizon.

Typically, the multi-armed bandit problem is studied under one of two assumptions:

- 1. The underlying reward distribution for each arm is fixed but unknown, and a prior of this distribution is specified as input (*stochastic multi-armed bandits* [2, 10, 42, 48]); or
- 2. The underlying rewards can vary with time in an adversarial fashion, and the comparison is against an optimal strategy that always plays one arm, albeit with the benefit of hindsight (*adversarial multi-armed bandits* [7, 15, 17]).

Relaxing both the assumptions simultaneously leads to the notorious restless bandit problem in decision theory, which in its ultimate generality, is PSPACE hard to even approximate [41]. In the last two decades, in spite of the growth of approximation algorithms and the numerous applications of restless bandits [3, 19, 20, 21, 37, 40, 13, 49, 52], the approximability of these have remained unexplored. In this paper, we provide a general algorithmic technique that yields the first O(1) approximations to a large class of these problems that are commonly studied in practice.

An important subclass of restless bandit problems are situations where the system is agnostic of the exploration – or the exploration gives us information about the state of the system but does not interfere with the evolution of the system. One such problem is the FEEDBACK MAB which models *opportunistic multi-channel access* at a wireless node [25, 30, 54]: The bandit corresponds to a wireless node with access to multiple noisy channels (arms). The state of the arm is the state (good/bad) of the channel, which varies according to a bursty 2-state Markov process. Playing the arm corresponds to transmitting on the channel, yielding reward if the transmission is successful (good channel state), and at the same time revealing to the transmitter the current state of the channel. This corresponds to the Gilbert-Elliot model [33] of channel evolution. The goal is to find a transmission policy of choosing one channel to transmit on every time step, that maximizes the long-term transmission rate. FEEDBACK MAB also models Unmanned Aerial Vehicle (UAV) routing [40]: the arms are locations of possibly interesting events, and whether a location is interesting or uninteresting follows a 2-state Markov processes. Visiting a location by the UAV corresponds to playing the arm, and yields reward if an interesting event is detected. The goal is to find a routing policy that maximizes the long-term average reward from interesting events.

This problem is also a special case of *Partially Observable Markov Decision Processes* or POMDPs [31, 44, 45]. The state of each arm evolves according to a Markov chain whose state is only observed when the arm is played. The player's partial information, encapsulated by the last observed state and the number of steps since last playing, yields a belief on the current state. (This belief is simply a probability distribution for the arm being good or bad.) The player uses this partial information in making the decision about which arm to play next, which in turn affects the information at future times. While such POMDPs are widely used in control theory, they are in general notoriously intractable [10, 31]. In this paper we provide the first O(1) approximation for the FEEDBACK MAB and a number of its important extensions. This represents the first approximation guarantee for a POMDP, and the first guarantee for a MAB problem with time-varying rewards that compares to an optimal solution allowed to switch arms at will.

Before we present the problem statements formally, we survey literature on the stochastic multi-armed bandit problem. (We discuss adversarial MAB after we present our model and results.)

1.1 Background: Stochastic MAB and Restless Bandits

The stochastic MAB was first formulated by Arrow *et al* [2] and Robbins [42]. It resides under a Bayesian (or decision theoretic) setting: we successively choose between several options given some prior information (specified by distributions), and our beliefs are updated via Bayes' rule conditioned on the results of our choices (observed rewards).

More formally, we are given a "bandit" with n independent arms. Each arm i can be in one of several states belonging to the set \mathcal{S}_i . At any time step, the player can play one arm. If arm i in state $k \in \mathcal{S}_i$ is played, it transitions in a Markovian fashion to state $j \in \mathcal{S}_i$ w.p. q_{kj}^i , and yields reward $r_k^i \geq 0$. The states of arms that are not played stay the same. The initial state models the prior knowledge about the arm. The states in general capture the posterior conditioned on the observations from sequential plays. The problems is, given the initial states of the arms, find a *policy* for playing the arms in order to maximize one of the following infinite horizon quantities: $\sum_{t=0}^{\infty} R_t \beta^t$ (discounted reward), or $\lim_{t\to\infty} \frac{1}{t} \sum_{t=0}^{\infty} R_t$ (average reward), where R_t is the expected reward of the policy at time step t and $\beta \in (0,1)$ is a discount factor. A *policy* is a (possibly implicit) specification of *fixing up front* which arm (or distribution over arms) to play for every possible joint state of the arms.

It is well-known that Bellman's equations [10] yield the optimal policy by dynamic programming. The main issue in the stochastic setting is in *efficiently computing and succinctly specifying* the optimal policy: The input to an algorithm specifies the rewards and transition probabilities for each arm, and thus has size linear in n, but the state space is exponential in n. We seek polynomial-time algorithms (in terms of the input size) that compute (near-) optimal policies with poly-size specifications. Moreover, we require the policies to be executable each step in poly-time.

Note that since a policy is a fixed (possibly randomized) mapping from the *exponential size* joint state space to a set of actions, ensuring poly-time computation and execution often requires simplifying the description of the optimal policy using the problem structure. The stochastic MAB problem is the most well-known decision problem for which such a structure is known: The *optimal* policy is a greedy policy termed the GITTINS index policy [18, 46, 10]. In general, an index policy specifies a single number called "index" for each state $k \in S_i$ for each arm i, and at every time step, plays the arm whose current state has the highest index. Index policies are desirable since they can be compactly represented, so they are the heuristic method of choice for several MDP problems. In addition, index policies are also optimal for several generalizations of the stochastic MAB, such as arm-acquiring bandits [51] and branching bandits [50]. In fact, a general characterization of problems for which index policies are optimal is now known [12].

Restless Bandits. In the stochastic MAB problem, the underlying reward distributions for each arm are fixed but unknown. However, if the rewards can vary with time, the problem stops admitting optimal index policies or efficient solutions. The problem now needs to be modeled as a *restless bandit* problem, first proposed by Whittle [52]. The problem statement of the restless bandits is similar to stochastic MAB, except that when arm i in state $k \in S_i$ is not played, it's state evolves to $j \in S_i$ with probability \tilde{q}_{kj}^i . Therefore, the state of each arm varies according to an *active* transition matrix q when the arm is played, and according to a *passive* transition matrix \tilde{q} if the arm is not played. Unlike the stochastic MAB problem, which is interesting only in the discounted reward setting¹, the restless bandit problem is interesting even in the *infinite horizon average reward* setting – this is the setting in which the problem has been typically studied, and so we limit ourselves to this setting in this paper. It is relatively straightforward to show that no index policy can be optimal for these problems; in fact, Papadimitriou and Tsitsiklis [41] show that for n arms, even when all q and \tilde{q} values are either 0 or 1 (deterministic transitions), computing the optimal policy is a PSPACE-hard problem. Their proof in fact shows that deciding if the optimal reward is non-zero is also

¹Playing the arm with the highest long-term average reward exclusively is the trivial optimal policy for stochastic MAB in the infinite-horizon average reward setting.

PSPACE-hard, hence ruling out any approximation algorithm as well.

On the positive side, Whittle [52] presents a poly-size LP relaxation of the problem. In this relaxation, the constraint that exactly one arm is played per time step is replaced by the constraint that one arm *on average* is played per time step. In the LP, this is the only constraint connecting the arms. (Such decision problems have been termed *weakly coupled* systems [29, 1].) Based on the Lagrangean of this relaxation, Whittle [52] defines a heuristic index that generalizes the Gittins index. This is termed the *Whittle Index* (see Section 3). Though this index is widely used in practice and has excellent empirical performance [3, 19, 20, 21, 37, 40, 49], the known theoretical guarantees ([49, 19]) are very weak. In summary, despite being very well-motivated and extensively studied, there are almost no positive results on approximation guarantees for the restless bandit problems.

1.2 Results and Roadmap

We provide the first approximation algorithm for both a restless bandit problem and a partially observable Markov decision problem by providing a $2 + \epsilon$ -approximate index policy for the FEEDBACK MAB problem which belongs to both classes. We show several other results; however, before presenting the specifics, we place our contribution in the context of existing techniques in control theory.

Technical Contributions. Our algorithmic technique for this problem (Section 2) involves solving (in polynomial time) the Lagrangean of Whittle's LP relaxation for a suitable (and subtle) "balanced" choice of the Lagrange multiplier, converting this into a feasible index policy, and using an amortized accounting of the reward for the analysis. We show that this technique is closely related to the Whittle index [52, 40, 30], and in fact, provide the first approximation analysis of (a subtle variant of) the Whittle index which is widely used in control theory literature in the context of FEEDBACK MAB problems (Section 3). We believe that analyzing the performance guarantees of the numerous indices used in the literature will increase and our analysis will provide an useful template.

However, the key difference between Whittle's index and our index policy is the following: The former chooses one Lagrange multiplier (or index) per state of each arm, with the policy playing the arm with the largest index. This has the advantage of separate efficient computations for different arms; and in addition, such a policy (the Gittins index policy [18]) is known to be optimal for the stochastic MAB. However, it is well-known [4, 8, 14] that this intuition about playing the arm with the largest index being optimal becomes increasingly invalid when complicated side-constraints such as time-varying rewards (FEEDBACK MAB), blocking plays, and switching costs are introduced. In fact, we show a concrete problem in Section 8 where the Whittle index has a $\Omega(n)$ performance gap.

In contrast to the Whittle index, our technique chooses a *single global* Lagrange multiplier via a careful accounting of the reward, and develops a feasible policy from it. Unlike the Whittle index, this technique is sufficiently robust to encompass a large number of often-used variants of FEEDBACK MAB problems: Plays with varying duration (Section 5), switching costs (Section 6), and observation costs (Section 7). In fact, we identify a general MONOTONE condition in restless bandit problems under which our technique applies (Section 4). Furthermore, our technique provides O(1) approximations to other classic restless bandit problems even when Whittle's index is polynomially sub-optimal: We show an example in the non-preemptive machine replenishment problem (Section 8). Finally, since our technique is based on solving the Lagrangean² (just like the Whittle index), the computation time is comparable to that for such indices.

In summary, our technique succeeds in finding the first provably approximate policies for widely-studied control problems, without sacrificing efficiency in the process. We believe that the generality of this tech-

²This aspect is explicit in Sections 2 and 3. However, in Sections 4–8, we have presented our algorithm in terms of first solving a linear program. However, it is easy to see that this is equivalent to solving the Lagrangean, and hence to the computation required for Whittle's index. The details are quite standard and can be reconstructed from those in Sections 2 and 3.

nique will be useful for exploring other useful variations of these problems as well as providing an alternate algorithm for practitioners.

Specific Results. In terms of specific results, the paper is organized as follows:

- We begin by presenting a $2 + \epsilon$ -approximation for FEEDBACK bandits in Section 2. We also provide a e/(e-1) integrality gap instance showing that out analysis is nearly tight.
- In Section 3 we show that our analysis technique can be used to prove that a thresholded variant of the Whittle index is a 2 approximation. We also show instances where the reward of any index policy is at least $1 + \Omega(1)$ factor from the reward of the optimal policy. Therefore although the Whittle index is not optimal, our result sheds light on its observed superior performance in this specific context.
- In Section 4 we generalize the result in Section 2 to define a general sub-class of restless bandit problems based on a critical set of properties: Separability and monotonicity. For this subclass, termed MONOTONE bandits (which generalizes FEEDBACK MAB), we provide a 2 approximation by generalizing the technique in Section 2. Our technique now introduces a balance constraint in the dual of the natural LP relaxation, and constructs the index policy from the optimal dual solution. We further show that in the absence of monotonicity or separability, the problem is either NP-Hard to approximate, or has unbounded integrality gap respectively.
- In Section 5 we extend FEEDBACK MAB (as well as MONOTONE bandits) to consider multiple simultaneous blocking plays of varying durations.
- In Section 6 we extend FEEDBACK MAB (and MONOTONE bandits) to consider switching costs.
- In Section 7 we extend FEEDBACK MAB to a variant where the information acquisition is varied, namely, an arm has to be explicitly probed at some cost to obtain its state.
- In Section 8, we derive a 2-approximation for a classic, restless bandit problem called non-preemptive machine replenishment [10, 23, 39]. We also show that the Whittle Index for this problem has a $\Omega(n)$ factor worse performance compared to the optimal policy. Thus the technique introduced in this paper can be superior to Whittle index or similar policies.

1.3 Related Work

Contrast with the Adversarial MAB Problem. While our problem formulations are based on the stochastic MAB problem, one might be interested in a formulation based on the adversarial MAB [7, 34]. Such a formulation might be to assume that rewards can vary adversarially, and that the objective is to compete with a restricted optimal solution that always plays the same arm but with the benefit of hindsight.

These different formulations result in fundamentally different problems. Under our formulation, the difficulty is computational: we want to compute *policies* for playing the arms, assuming stochastic models of how the system varies with time. under the adversarial formulation, the difficulty is *informational*: we would be interested in the *regret* of not having the benefit of hindsight. A sequence of papers show near-tight regret bounds in fairly general settings [5, 6, 7, 15, 17, 34, 36]. However, applying this framework is not satisfying: It is straightforward to show that a policy for FEEDBACK MAB that is allowed to switch arms can be $\Omega(n)$ times better than a policy that is not allowed to do so (even assuming hindsight). Another approach would be to define each policy as an "expert", and use the low-regret experts algorithm [15]; however, the number of policies is super-exponentially large, which would lead to weak regret bounds, along with exponential-size policy descriptions and exponential per-step execution time.

We note that developing regret bounds in the presence of changing environments has received significant interest recently in computational learning [5, 7, 16, 32, 43]; however, this direction requires strong assumptions such as bounded switching between arms [5] and slowly varying environments [32, 43], both of which assumptions are inapplicable to FEEDBACK MAB. In independent work, Slivkins and Upfal [43] consider the modification of FEEDBACK MAB where the underlying state of the arms vary according to a reflected Brownian motion with bounded variance. As discussed in [43], this problem is technically very different from ours, even requiring different performance metrics.

Other Related Work. The results in [24, 22, 26, 27] consider variants of the stochastic MAB where the underlying reward distribution does not change and only a limited time is allotted to learning about this environment. Although several of these results use LP rounding, they have little connection to the duality based framework considered here.

Our duality based framework shows a 2-approximate index policy for non-preemptive machine replenishment (Section 8). Elsewhere, Munagala and Shi [39] considered the special case of preemptive machine replenishment problem, for which the Whittle index is equivalent to a simple greedy scheme. They show that this greedy policy, though not optimal, is a 1.51 approximation. However, the techniques there are based on queuing analysis, and do not extend to the non-preemptive case where the Whittle index can be an arbitrarily poor approximation (as shown in Section 8).

Our solution technique differs from primal-dual approximation algorithms [47] and online algorithms [53], which relax either the primal or the dual complementary slackness conditions using a careful dual-growing procedure. Our index policy and associated potential function analysis crucially exploit the structure of the *optimal* dual solution that is gleaned using both the *exact* primal as well as dual complementary slackness conditions. Furthermore, our notion of dual balancing is very different from that used by Levi *et al* [35] for designing online algorithms for stochastic inventory management.

2 The FEEDBACK MAB Problem

In this problem, first formulated independently in [25, 54, 30, 40], there is a bandit with n independent arms. Arm i has two states: The good state g_i yields reward r_i , and the bad state b_i yields no reward. The evolution of state of the arm follows a bursty 2-state Markov process which does not depend on whether the arm is played or not at a time slot. Let s_{it} denote the state of arm i at time t. Denote the transition probabilities of the Markov chain as follows: $\Pr[s_{i(t+1)} = g_i | s_{it} = b_i] = \alpha_i$ and $\Pr[s_{i(t+1)} = b_i | s_{it} = g_i] = \beta_i$. The α_i, β_i, r_i values are specified as input. The "burstiness" assumption simply means $\alpha_i + \beta_i \leq 1 - \delta$ for some small $\delta > 0$ specified as part of the input. The evolution of states for different arms are independent. Any policy chooses at most one arm to play every time slot. Each play is of unit duration, yields reward depending on the state of the arm, and reveals to the policy the current state of that arm. When an arm is not played, the true underlying state cannot be observed, which makes the problem a POMDP. The goal is to find a policy to play the arms in order to maximize the infinite horizon average reward.

First observe that we can change the reward structure of FEEDBACK MAB so that when an arm is played, we obtain reward from the *last-observed* state instead of the currently observed state. This does not change the average reward of any policy. This allows us to encode all the state of each arm as follows.

Proposition 2.1. From the perspective of any policy, the state of any arm can be encoded as (s,t), which denotes that it was last observed $t \ge 1$ steps ago to be in state $s \in \{g_i, b_i\}$.

Note that any policy maps each possible joint state of n arms into an action of which arm to play. Such a mapping has size exponential in n. The standard heuristic is to consider index policies: Policies which define an "index" or number for each state (s_i, t) and play the arm with the highest current index. The

following theorem shows that playing the arm with the highest myopic reward does not work, and that index policies in general are non-optimal. Therefore, our problem is interesting and the best we can hope for with index policies is a O(1) approximation.

Theorem 2.1. (Proved in Appendix A) For FEEDBACK MAB, the reward of the optimal policy has an $\Omega(n)$ gap against that of the myopic index policy and an $\Omega(1)$ gap against that of the optimal index policy.

Roadmap. In this section, we show that a simple index policy is a $(2+\epsilon)$ approximation. This is based on a natural LP relaxation suggested by Whittle which we discuss in Section 2.1; this formulation will have infinitely many constraints. We then consider the Lagrangean of this formulation in Section 2.2, and analyze its structure via duality, which enables computing its optimal solution in polynomial time. At this point, we deviate significantly from previous literature, and present our main contribution in Section 2.3: A subtle and powerful "balanced" choice of the Lagrange multiplier, which enables the design of an intuitive index policy, BALANCEDINDEX, along with an equally intuitive analysis. We use duality and potential function arguments to show that the policy is $(2+\epsilon)$ approximation. We conclude by showing that the gap of Whittle's relaxation is $e/(e-1)\approx 1.58$, indicating that our analysis is reasonably tight. This analysis technique generalizes easily (explored in Sections 4-8) and has rich connections to other index policies, most notably the Whittle index (explored in Section 3).

2.1 Whittle's LP

Whittle's LP is obtained by effectively replacing the hard constraint of playing one arm per time step, with allowing multiple plays per step but requiring one play per step *on average*. Hence, the LP is a *relaxation* of the optimal policy.

Definition 1. Let v_{it} be the probability of the arm i being in state g_i when it was last observed in state b_i exactly t steps ago. Let u_{it} be the same probability when the last observed state was g_i . We have:

$$v_{it} = \frac{\alpha_i}{\alpha_i + \beta_i} (1 - (1 - \alpha_i - \beta_i)^t) \quad and \quad u_{it} = \frac{\alpha_i}{\alpha_i + \beta_i} + \frac{\beta_i}{\alpha_i + \beta_i} (1 - \alpha_i - \beta_i)^t$$

Fact 2.2. The functions v_{it} and $1 - u_{it}$ are monotonically increasing and concave functions of t.

We now present Whittle's LP, and interpret it in the lemma that immediately follows.

Lemma 2.3. The optimal objective to Whittle's LP, OPT, is at least the value of the optimal policy.

Proof. Consider the optimal policy. In the execution of this policy, for each arm i and state (s,t) for $s \in \{g,b\}$, let the variable x^i_{st} denote the probability (or fraction of time steps) of the event: Arm i is in state (s,t) and gets played. Let y^i_{st} correspond to the probability of the event that the state is (s,t) and the arm is

not played. Since the underlying Markov chains are ergodic, the optimal policy when executed is ergodic, and the above probabilities are well-defined.

Now, at any time step, some arm i in state (s,t) is played, which implies the x_{st}^i values are probabilities of mutually exclusive events. This implies they satisfy the first constraint in the LP. Similarly, for each arm i, at any step, this arm is in some state (s,t) and is either played or not played, so that the x_{st}^i, y_{st}^i correspond to mutually exclusive events. This implies that for each i, they satisfy the second constraint. For any arm i and state (s,t), the LHS of the third constraint is the probability of being in this state, while the RHS is the probability of entering this state; these are clearly identical in the steady state. For arm i, the LHS of the fourth (resp. fifth) constraint is the probability of being in state (g,1) (resp. (b,1)), and the RHS is the probability of entering this state; again, these are identical.

This shows that the probability values defined for the execution of the optimal policy are feasible for the constraints of the LP. The value of the optimal policy is precisely $\sum_{i=1}^{n}\sum_{t\geq 1}r_{i}x_{gt}^{i}$, which is at most OPT – the maximum possible objective for the LP.

The above LP encodes in one variable x_{st}^i the probability the arm i is in state (s,t) and gets played; however, we note that in the optimal policy, this decision to play actually depends on the joint state of all arms. This separation of the joint probabilities into individual probabilities effectively relaxes the condition of having one play per step, to allowing multiple plays per step but requiring one play per step on average. While the policy generated by Whittle's LP is infeasible, the relaxation allows us to compute an upper-bound on the value of the optimal feasible policy.

We note $y_{st}^i = \sum_{t'>t} x_{st'}^i$. It is convenient to eliminate the variables y_{st}^i by substitution and the last two constraints collapse into the same constraint. Thus, we have the natural LP formulation shown in Figure 1. We note that the first constraint can either be an inequality (\leq) or an equality; w.l.o.g., we use equality, since we can add a dummy arm that does not yield any reward on playing.

Figure 1: The linear program (WHITTLE) for the FEEDBACK MAB problem.

From now on, let OPT denote the value of the optimal solution to (WHITTLE). The LP in its current form has infinitely many constraints; we will now show that this LP can be solved in polynomial time to arbitrary precision by finding structure in the Lagrangean.

2.2 Decoupling Arms via the Lagrangean

In (WHITTLE), the only constraint connecting different arms is the constraint:

$$\sum_{i=1}^{n} \sum_{s \in \{q,b\}} \sum_{t \ge 1} x_{st}^{i} = 1$$

We absorb this constraint into the objective via Lagrange multiplier $\lambda \geq 0$ to obtain the following objective:

Max.
$$\lambda + G(\lambda) \equiv \lambda + \sum_{i=1}^{n} \sum_{t>1} \left(r_i x_{gt}^i - \lambda \left(x_{gt}^i + x_{bt}^i \right) \right)$$
 (LPLAGRANGE(λ))

$$\begin{array}{cccc} \sum_{t \geq 1} \sum_{s \in \{g,b\}} t x_{st}^{i} & \leq & 1 & \forall i \\ \sum_{t \geq 1} x_{bt}^{i} v_{it} & = & \sum_{t \geq 1} x_{gt}^{i} (1 - u_{it}) & \forall i \\ x_{st}^{i} & \geq & 0 & \forall i, s \in \{g,b\}, t \end{array}$$

Through the Lagrangean, we have effectively removed the only constraint that connected multiple arms. LPLAGRANGE(λ) now yields n disjoint maximization problems, one for each arm i: At any time step, arm i can be played (and reward obtained from it), or not played. Whenever the arm is played, we incur a penalty λ in addition to the reward. The goal is to maximize the expected reward minus cost. Note that if the penalty is zero, the arm is played every step, and if the penalty is sufficiently large, the optimal solution would be to never play the arm.

Definition 2. For each arm i, let $L_i(\lambda)$ denote the optimal policy, and let $H_i(\lambda)$ denote the optimal reward minus penalty. Note that the global reward minus penalty is the sum for each arm: $G(\lambda) = \sum_{i=1}^{n} H_i(\lambda)$.

2.2.1 Characterizing the Optimal Single-arm Policy

We first show that the optimal policy $L_i(\lambda)$ for any arm i belongs to the class of policies $\mathcal{P}_i(t)$ for $t \geq 1$, whose specification is presented in Figure 2. Intuitively, step (1) corresponds to exploitation, and step (2) to exploration. Set $\mathcal{P}_i(\infty)$ to be the policy that never plays the arm.

Policy $\mathcal{P}_i(t)$:

- 1. If the arm was just observed to be in state g, then play the arm.
- 2. If the arm was just observed to be in state b, wait t-1 steps and play the arm.

Figure 2: The Policy $\mathcal{P}_i(t)$.

To show this, we take with the dual of LPLAGRANGE(λ):

$$\begin{array}{lll} \text{Minimize} & \lambda + \sum_{i=1}^n h_i & \text{Whittle-Dual}(\lambda) \\ & \lambda + th_i & \geq & r_i - (1-u_{it})p_i & \forall i,t \geq 1 \\ & \lambda + th_i & \geq & v_{it}p_i & \forall i,t \geq 1 \\ & h_i & \geq & 0 & \forall i \end{array}$$

The fact that the optimal single arm policy $L_i(\lambda)$ belongs to the class $\{P_i(t)\}$ comes from (5) of the following lemma.

Lemma 2.4. For any $\lambda \geq 0$, in the optimal solution to WHITTLE-DUAL (λ) , for any arm with $h_i > 0$:

- 1. $h_i = H_i(\lambda)$.
- 2. $p_i > 0$.
- 3. For some $t_i \geq 1$, $x_{bt_i}^i > 0$ and $\lambda + t_i h_i = v_{it_i} p_i$.
- 4. $x_{a1}^{i} > 0$ and $\lambda + h_{i} = r_{i} \beta p_{i}$.
- 5. The optimal single-arm policy for arm i is $L_i(\lambda) = \mathcal{P}_i(t_i)$.

Proof. The first part follows the definition of strong duality. The problem LPLAGRANGE(λ), ignoring the constant λ in the objective, separates into n separate LPs, one for each arm. The dual objective for arm i is precisely h_i , which must be the same as the primal objective, $H_i(\lambda)$.

If $h_i = H_i(\lambda) > 0$, the solution to the LP for arm i is the policy $L_i(\lambda)$. In order to have non-zero $H_i(\lambda)$, such a policy must play the arm first in some state (b, t_i) and state (g, t_i') . Since x_{st}^i is the probability this policy plays in state (s, t), this implies $x_{bt_i}^i > 0$ and $x_{qt_i'}^i > 0$.

Since $x_{bt_i}^i > 0$, by complementary slackness, we have $\lambda + t_i h_i = v_{it_i} p_i$. Since the LHS is at least zero, this implies $p_i \ge 0$. This proves parts (2) and (3).

To see part (4), observe that for the set of constraints $\lambda + th_i \ge r_i - (1 - u_{it})p_i$, since $1 - u_{it}$ is a monotonically increasing function of t, the RHS is monotonically decreasing in t. Since the LHS is monotonically increasing, if the LHS and RHS are equal, they have to be so for t = 1. Now, since $x_{gt'_i}^i > 0$, by complementary slackness, $\lambda + t'_i h_i = r_i - (1 - u_{it'_i})p_i$. By the above argument, $t'_i = 1$, which completes the proof of part (4).

Since $x_{g1}^i > 0$ and $x_{bt_i}^i > 0$, the optimal policy for $L_i(\lambda)$ plays the arm in state (g,1) and in state (b,t_i) , which is precisely the description of $\mathcal{P}_i(t_i)$. This proves part (5).

It will be instructive to interpret the problem $L_i(\lambda)$ as follows: Amortize the reward so that for each play, the arm i yields a steady reward of λ . The goal is to find the single-arm policy that optimizes the excess reward per step over and above the amortized reward λ per play. As we have shown above, the optimal value for this problem is precisely $H_i(\lambda)$, and the policy $L_i(\lambda)$ that achieves this belongs to the class $\{\mathcal{P}_i(t), t \geq 1\}$.

2.2.2 Solving LPLAGRANGE(λ)

Having decomposed the program LPLAGRANGE(λ) into independent maximization problems for each arm, and having characterized the optimal single-arm policies, we can now solve the program in polynomial time. It will turn out this can be solved by simple function maximization via closed form expressions.

Definition 3. For policy $\mathcal{P}_i(t)$, let $R_i(t)$ denote the expected per-step reward, and let $Q_i(t)$ denote the expected rate of play. Let $F_i(\lambda, t) = R_i(t) - \lambda Q_i(t)$ denote the value of $\mathcal{P}_i(t)$. Also define:

$$t_i(\lambda) = \operatorname{argmax}_{t>1} F_i(\lambda, t) = \operatorname{argmax}_{t>1} R_i(t) - \lambda Q_i(t)$$
 (1)

Finally, let $R_i(\lambda) = R_i(t_i(\lambda))$ and $Q_i(\lambda) = Q_i(t_i(\lambda))$

Note that the optimal reward minus cost for arm i is simply $H_i(\lambda) = \max_{t \geq 1} R_i(t) - \lambda Q_i(t) = R_i(t_i) - \lambda Q_i(t_i)$. Since each $\mathcal{P}_i(t)$ corresponds to a Markov Chain, it is straightforward to obtain closed form expressions for $R_i(t)$ and $Q_i(t)$.

Lemma 2.5. In playing an arm with reward r, transition probabilities α and β , the policy $\mathcal{P}(t)$ yields average reward $R(t) = r \frac{v_t}{v_t + t\beta}$, and expected rate of play $Q(t) = \frac{v_t + \beta}{v_t + t\beta} \geq \frac{1}{t}$. Recall that $v_t = \frac{\alpha}{\alpha + \beta}(1 - (1 - \alpha - \beta)^t)$ is the probability the arm is good given it was observed to be bad t steps ago.

Proof. The Markov chain describing the policy $\mathcal{P}(t)$ is shown in Figure 3, and has t+1 states which we denote $s,0,1,2,\ldots,t-1$. The state s corresponds to the arm being observed to be in state g, and the state j corresponds to the arm being observed in state b exactly j steps ago. The transition probability from state j to state j+1 is 1, from state s to state s is s, from state s is s, and from state s to state s is s, and from state s to state s is s, and from state s to state s is s, and from state s to state s is s, and from state s to state s is s, and from state s to state s is s, and from state s to state s is s, and from state s to state s is s, and from state s to state s is s, and from state s is s, and from state s.

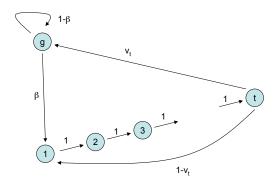


Figure 3: Markov Chain for policy $\mathcal{P}(t)$.

 $\pi_s + t\pi_0 = 1$. Furthermore, by considering transitions into and out of s, we obtain: $\beta \pi_s = v_t \pi_{t-1} = v_t \pi_0$. Combining these, we obtain: $\pi_s = \frac{v_t}{v_t + t\beta}$, and $\pi_0 = \frac{\beta}{v_t + t\beta}$. Now we have:

$$R(t) = r[(1-\beta)\pi_s + v_t\pi_0] = r\pi_s = r\frac{v_t}{v_t + t\beta}$$
$$Q(t) = \pi_s + \pi_{t-1} = \frac{v_t + \beta}{v_t + t\beta}$$

Lemma 2.6. (Proved in Appendix A) For each arm i, the optimal reward minus penalty of the single arm policy for arm i is

$$H_i(\lambda) = \max_{t \ge 1} F_i(\lambda, t) = \max_{t \ge 1} \left(\frac{(r_i - \lambda)v_{it} - \lambda\beta_i}{v_{it} + t\beta_i} \right)$$

The maximum value $t_i(\lambda) = argmax_{t>1} F_i(\lambda, t)$ satisfies the following:

1. If
$$\lambda \geq r_i \left(\frac{\alpha_i}{\alpha_i + \beta_i(\alpha_i + \beta_i)} \right)$$
, then $t_i(\lambda) = \infty$, and $H_i(\lambda) = 0$.

2. If $\lambda = r_i \left(\frac{\alpha_i}{\alpha_i + \beta_i(\alpha_i + \beta_i)} \right) - \rho$ for some $\rho > 0$, then $t_i(\lambda)$ (and hence $H_i(\lambda)$) can be computed in time polynomial in the input size and in $\log(1/\rho)$ by binary search.

2.3 The BALANCEDINDEX Policy

Though we could now use LPLAGRANGE(λ) to solve Whittle's LP by finding the λ so that $\sum_{i=1}^n Q_i(\lambda) \approx 1$ (refer Appendix A.3 for details), our 2-approximation policy will *not* be based this approach. For our analysis to work, we must make a subtle but crucial modification: We will instead set λ to be the sum of the excess reward for all single-arm policies $\sum_{i=1}^n H_i(\lambda)$. (Recall that we can interpret λ to be a penalty per play, so in the optimal single-arm policy for arm i, $H_i(\lambda)$ is the average reward minus penalty.) Note that by Lemma 2.7, this implies $\lambda \geq OPT/2$ and $\sum_{i=1}^n H_i(\lambda) \geq OPT/2$. Intuitively, we are forcing the Lagrangean to *balance* short-term reward (represented by λ) with long-term average reward (represented by $\sum_{i=1}^n H_i(\lambda)$). Our *balance* technique can be generalizes to many other restless bandit problems (see Sections 4 – 8).

We first show how to compute this value of λ in polynomial time. We begin by presenting the connection between $G(\lambda = \sum_{i=1}^{n} H_i(\lambda))$ and OPT, the value of the optimal solution to (WHITTLE).

Lemma 2.7. For any λ , we have: $\lambda + G(\lambda) = \lambda + \sum_{i=1}^{n} H_i(\lambda) \geq OPT$.

Proof. By Lemma 2.4, part (1), we have: $\lambda + \sum_{i=1}^{n} H_i(\lambda) = \lambda + \sum_{i=1}^{n} h_i$. The latter is the objective of the dual of (WHITTLE), which implies the lemma by weak duality.

Lemma 2.8. $h_i = H_i(\lambda)$ is a non-increasing function of λ .

Proof. Recall from Lemma 2.4 that $h_i = H_i(\lambda)$. For any λ , consider the value $F_i(\lambda,t) = R_i(t) - \lambda Q_i(t)$ of the policy $\mathcal{P}_i(t)$. Since this decreases as λ increases, $H_i(\lambda) = \max_t F_i(\lambda,t)$ is also a non-increasing function of λ .

Lemma 2.9. In polynomial time, we can find a λ so that $\lambda \geq (1 - \epsilon)OPT/2$, and $G(\lambda) = \sum_{i=1}^{n} H_i(\lambda) = \sum_{i=1}^{n} h_i \geq OPT/2$.

Proof. First note by Lemma 2.8 that $G(\lambda) = \sum_{i=1}^n H_i(\lambda)$ is monotonically non-increasing in λ . Therefore, start with $\lambda = \sum_{i=1}^n r_i$, and λ scale down by by a factor of $(1+\epsilon)$ until $\lambda < G(\lambda)$. Note that for any λ , the value of $G(\lambda)$ can be computed in poly-time by Lemma 2.6. At this point, let $\lambda' = \lambda(1+\epsilon)$. Since $G(\lambda') \leq \lambda'$, by Lemma 2.7, we have $\lambda' \geq OPT/2$, which implies $\lambda \geq (1-\epsilon)OPT/2$. Further, since $\lambda < G(\lambda)$, again by Lemma 2.7, we have $G(\lambda) \geq OPT/2$.

2.3.1 Index Policy

We start with the value of λ from Lemma 2.9. The policy only works with the subset of arms S so that for $i \in S$, we have $H_i(\lambda) > 0$. For this λ , the solution to LPLAGRANGE(λ) yields one policy $\mathcal{P}_i(t_i(\lambda))$ of value $H_i(\lambda)$ for each arm $i \in S$ (see Lemma 2.4). Let $t_i = t_i(\lambda)$. Recall that if an arm was last observed in state $s \in \{g,b\}$ some $t \geq 1$ steps ago, then its state is denoted (s,t). We call an arm i in state (g,1) as good; in state (b,t) for $t \geq t_i$ as ready, and in state (b,t) for $t < t_i$ as bad. The policy is shown in Figure 4.

BALANCEDINDEX Policy for FEEDBACK MAB

Consider only arms with $H_i(\lambda) > 0$; denote these as set S.

Play arms in S according to the following priority scheme:

- 1. **Exploit:** Play any arm in state (g, 1) (good state):
- 2. If condition (1) does not hold:
 - (a) **Explore:** Play any arm $i \in S$ in state (b, t) such that $t \ge t_i$ (ready state).
 - (b) **Idle:** If no arm is *good* or *ready* (all arms *bad*), do not play at this step.

Figure 4: The BALANCEDINDEX Policy for FEEDBACK MAB.

Note that the way the scheme works, at most one arm can be in state (g, 1) at any time step, and if such an arm exists, this arm is played at the current step (and in the future until it switches out of this state). The above can be thought of as executing the policies $\mathcal{P}_i(t_i)$ for arms $i \in S$ independently and in case of simultaneous attempts to play, resolving conflicts according to the above priority scheme.

Though the above policy is not written as an index policy, it is equivalent to the following index: There is a dummy arm with index 0 that does not yield reward on playing. If $h_i = H_i(\lambda) = 0$, the index for all states of this arm is -1. For arms with $h_i > 0$, the index for state (g,1) is 2; that for states (b,t) with $t \ge t_i$ is 1, and that for states (b,t) with $t < t_i$ is -1.

2.3.2 Analysis

We now prove that the BALANCEDINDEX policy is in fact a 2-approximation. The proof is based on the fact that the Lagragean λ and the excess rewards $h_i = H_i(\lambda)$ give us a way of accounting for the average reward. And by Lemma 2.7, $\lambda \geq OPT/2$ and $\sum h_i \geq OPT/2$, which gives us a way of linking the rewards from our policy to the LP optimum.

Theorem 2.10. The BALANCEDINDEX policy is a $2 + \epsilon$ approximation to FEEDBACK MAB. Furthermore, this policy can be computed in polynomial time.

Proof. Recall that the reward of optimal single arm policy $\mathcal{P}_i(t_i)$ is $R_i(\lambda) = H_i(\lambda) + \lambda Q_i(\lambda)$, so that this reward can be accounted as $H_i(\lambda) = h_i$ per step plus λ per play. We use this amortization of rewards to show that the average reward of our index policy is at least OPT/2.

Focus on any arm i, we call a step *blocked* for the arm if the arm is ready for play–the state is (b,t) where $t \geq t_i$ –but some other arm is played at the current step. Consider only the time steps which are *not* blocked for arm i. For these time steps, the arm behaves as follows: It is continuously played in state (g,1). Then it transitions to state (b,1) and moves in t_i-1 time steps to state (b,t_i-1) . After this the arm might be blocked, and the next state that is not blocked is (b,t) for some $t \geq t_i$, at which point the arm is played. Using the formula for R(t) from Lemma 2.5, and since $v_{it} \geq v_{it_i}$ for $t \geq t_i$, we have

$$R_i(t) \ge r_i \frac{v_{it}}{v_{it} + t_i \beta_i} \ge r_i \frac{v_{it_i}}{v_{it_i} + t_i \beta_i} = R_i(t_i)$$

which implies that the per-step reward of this single arm policy for arm i restricted to the non-blocked time steps is at least the per-step reward $R_i(t_i)$ of the optimal single-arm policy $\mathcal{P}_i(t_i)$. Therefore, for these non-blocked steps, the reward we get is at least $h_i = H_i(\lambda)$ per step, and at least λ per play.

Now, on steps where no arm is played, none of the arms is blocked by definition, so our amortization yields a per-step reward of at least $\sum_{i \in S} h_i \ge OPT/2$. On steps when some arm is played, the arm that is played by definition cannot not blocked, so we get a reward of at least $\lambda \ge (1 - \epsilon)OPT/2$ for this step. This completes the proof.

2.3.3 Alternate Analysis

The above analysis is very intuitive. We now present an alternative way to analyze the policy, that leads to a more generalizable technique. This uses a Lyapunov (potential) function argument. Recall from Lemma 2.4 that $h_i = H_i(\lambda)$; further that $t_i = t_i(\lambda)$. Define the potential Φ_i for each arm i at any time as follows:

Definition 4. If arm i moved to state b some y steps ago $(y \ge 1)$, the potential Φ_i is $h_i(\min(y, t_i) - 1)$. In the state g_i the potential is p_i . Recall that p_i is the optimal dual variable in WHITTLE-DUAL (λ) .

Let Φ_T denote the total potential, $\sum_{i=1}^n \Phi_i$, at any step T and let R_T denote the total reward accrued until that step. Define the function $\mathcal{L}_T = t \cdot OPT/2 - R_T - \Phi_T$. Let $\Delta R_T = R_{T+1} - R_T$ and $\Delta \Phi_T = \Phi_{T+1} - \Phi_T$.

Lemma 2.11. \mathcal{L}_T is a Lyapunov function. i.e., $\mathbf{E}[\mathcal{L}_{T+1}|\mathcal{L}_T] \leq \mathcal{L}_T$. Equivalently, at any step:

$$\mathbf{E}[\Delta R_T + \Delta \Phi_T | R_T, \Phi_T] \ge (1 - \epsilon)OPT/2$$

Proof. At a given step, suppose the policy does nothing, then all arms are "not ready". The total increase in potential is precisely

$$\Delta \Phi_T = \sum_{i \in S} h_i = G(\lambda) \ge OPT/2$$

On the other hand, suppose that the policy plays arm i, which has last been observed in state b and has been in that state for $y \ge t_i$ steps. With probability $q \ge v_{it_i}$ the observed state is g_i and the change in reward $\Delta R_T = r_i$ and the change in potential is $p_i - h_i(t_i - 1)$. With probability 1 - q the observed state is b and the change in potential is $-h_i(t_i - 1)$ (and there is no change in reward). Thus in this case since $q \ge v_{it_i}$, and $p_i \ge 0$, we have:

$$\mathbf{E}[\Delta R_T + \Delta \Phi_T | R_T, \Phi_T] \ge q p_i - h_i(t_i - 1) \ge v_{it_i} p_i - h_i(t_i - 1) \ge \lambda + h_i \ge (1 - \epsilon)OPT/2$$

The penultimate inequality follows from Lemma 2.4, part (3). Note that the potentials of arms not played cannot decrease, so that the first inequality is valid.

Finally supposing the policy plays an arm i which was last observed in state g_i and played in the last step, with probability $1 - \beta_i$ the increase in reward is r_i and the potential is unchanged. With probability β_i the potential will decrease by $-p_i$. Therefore in this case, by Lemma 2.4, part (4).

$$\mathbf{E}[\Delta R_T + \Delta \Phi_T | R_T, \Phi_T] \ge r_i - \beta_i p_i \ge \lambda + h_i \ge (1 - \epsilon)OPT/2$$

By their definition, the potentials Φ_T are bounded independent of the time horizon, by telescoping summation, the above lemma implies that $\lim_{T\to\infty}\frac{\mathbf{E}[R_T]}{T}\geq (1-\epsilon)OPT/2$. This proves Theorem 2.10.

Gap of Whittle's LP. The following theorem shows that our analysis is almost tight (considering that our 2-approximation is against Whittle's LP).

Theorem 2.12. (Proved in Appendix A) The gap of Whittle's LP is arbitrarily close to $e/(e-1) \approx 1.58$.

3 Analyzing the Whittle Index for FEEDBACK MAB

Before generalizing our 2-approximation algorithm to a larger subclass of restless bandit problems, we explore the connection between our analysis and the well-known Whittle Index used in practice. This section can be skipped without losing continuity of the paper.

A well-studied index policy for restless bandit problems is the Whittle Index [52]. In the context of FEEDBACK MAB, this index has been independently studied by Le Ny et al [40] and subsequently by Liu and Zhao [37]. Both these works give a closed form expressions for this index and show near-optimal empirical performance. Our main result in this section is to justify the empirical performance by showing that a simple but very natural modification of this index in order to favor myopic exploitation yields a 2-approximation. The modification simply involves giving additional priority to arms in state (g, 1) if their myopic expected next step reward $r_i(1 - \beta_i)$ is at least a threshold value.

3.1 Description of the Whittle Index

Defined in general, the Whittle's index for each state x is the largest penalty-per-play λ such that the optimal policy is indifferent between playing in x and not playing. In our specific problem, the current state for each arm i is captured by the tuple (s,t)- the arm was last seen to be $s \in \{g,b\}$ (good or bad) t steps ago. The Whittle index $\Pi_i(s,t)$ is a non-negative real numbers computed as follows: using the notation from Section 2.2., for any penalty per play λ , there is a single-arm policy $L_i(\lambda)$ that maximizes the average reward minus penalty (excess reward) $H_i(\lambda)$ over the infinite horizon. When $\lambda = \infty$, the optimal policy never plays; when $\lambda = 0$, the optimal policy would play in any state. As λ is decreased from ∞ , at some value λ^* , the decision in state (s,t) changes from "not play" to "play". The Whittle index $\Pi_i(s,t)$ is

Whittle Index Policy: Play the arm i whose current state (s,t) has the highest index $\Pi_i(s,t)$.

Figure 5: The Whittle Index Policy.

precisely this value of λ^* . The Whittle index policy always plays the arm with the highest Whittle's index (Fig. 5).

Remarks. The Whittle index is strongly decomposable, *i.e.*, can be computed separately for each arm. Further, we have defined λ as a penalty (or amortized reward) per play, while Whittle defines it as a reward for not playing (which he terms the *subsidy for passivity*); it is easy to see that both these formulations are equivalent. Finally, for FEEDBACK MAB, it can be shown [40, 37] that for any state (s,t), there is a *unique* $\lambda \in (-\infty, \infty)$ where the decision switches between "play" and "not play", *i.e.*, the decision is *monotone* in λ . Strictly speaking, the Whittle index is defined only for such systems (termed *indexable* by Whittle [52]); we will define this aspect away by insisting that the index λ^* is the *largest* value where a switch happens.

We present an explicit connection of Whittle's index to LPLAGRANGE(λ).

Lemma 3.1. (Proved in Appendix A) Recall the notation $L_i(\lambda)$ and $\mathcal{P}_i(t)$ from Section 2.2. The following hold for $\Pi_i(s,t)$:

- 1. $\Pi_i(s,t) \geq 0$ for all states (s,t) where $s \in \{g,b\}$ and $t \in \mathbf{Z}^+$.
- 2. $\Pi_i(g, 1) = r_i(1 \beta_i)$, and $\Pi_i(b, t) \leq \Pi_i(g, 1)$ for all $t \geq 1$.
- 3. $\Pi_i(b,t) = \max\{\lambda | L_i(\lambda) = \mathcal{P}_i(t)\}$, and is a monotonically non-decreasing function of t.

Though Whittle's index is widely used, it is not clear how to analyze it since it leads to complicated priorities between arms. We now show that our balancing technique also implies an analysis for a slight but non-trivial modification to Whittle's index.

3.2 The Threshold-Whittle Policy

We now show that modifying the index slightly to exploit the myopic next step reward in good states g yields a 2 approximation. Note that the myopic next step reward of an arm i in state g is precisely $\Pi_i(g,1)=r_i(1-\beta_i)$. The modification essentially favors exploiting such a "good" state if the myopic reward is at least a certain threshold value. In particular, we analyze the policy Threshold-Whittle(λ) shown in Figure 6, where we set $\lambda=\lambda^*$, where λ^* is the value where $\lambda^*=\sum_{i=1}^n H_i(\lambda^*)$ (refer Section 2.3).

```
\label{eq:theorem} \begin{array}{l} \textbf{Threshold-Whittle}(\lambda) \\ \textbf{At any time step:} \\ \textbf{If } \exists \mbox{ arm } i \mbox{ in state } (g,1) \mbox{ whose Whittle index is } \Pi_i(g,1) = r_i(1-\beta_i) \geq \lambda \\ \\ \textbf{then Play arm } i. \\ \textbf{else Play the arm with the highest Whittle index.} \end{array}
```

Figure 6: Policy THRESHOLD-WHITTLE(λ). It exploits arm i if the myopic reward in state (g, 1) is $\geq \lambda$.

Note that the above policy can be restated as playing the arm with the highest modified index, which is computed as follows: For arm i, if $\Pi_i(g,1) = r_i(1-\beta_i) \ge \lambda$, the modified index for state (g,1) is ∞ , else the modified index is the same as the Whittle index.

Theorem 3.2. THRESHOLD-WHITTLE(λ^*) is a 2 approximation for FEEDBACK MAB. Here, λ^* satisfies $\lambda^* = \sum_{i=1}^n H_i(\lambda^*)$ (refer Section 2.3).

3.3 Proof of Theorem 3.2

We now prove the above result by modifying our analysis of the BALANCEDINDEX policy (from Figure 4). Recall that S is the set of arms with $h_i > 0$ in the optimal solution to WHITTLE-DUAL (λ^*) . For such arms, $t = t_i$ is the first time instant when $\lambda + th_i \geq p_i v_{it}$ is tight. For arm $i \in S$, state (s,t) is good if s = g and t = 1; ready if s = b and $t \geq t_i$; and bad otherwise. The index policy from Figure 4 favors good over ready states, and does not play any arm in bad states.

Claim 3.3. For any arm i, exactly one of the following is true for WHITTLE-DUAL(λ^*) and LPLAGRANGE(λ^*).

- 1. The constraint $\lambda^* + th_i \ge v_{it}p_i$ is first tight at $t = t_i$. Then, $\Pi_i(b, t_i 1) < \lambda^*$ and $\Pi_i(b, t_i) \ge \lambda^*$. Further, $\Pi_i(q, 1) = r_i(1 \beta_i) \ge \lambda^*$ and $h_i > 0$.
- 2. The constraint $\lambda^* + th_i \geq v_{it}p_i$ is not tight for any t. Then, $\Pi_i(b,t) \leq \lambda^*$ for all $t \geq 1$, and $h_i = 0$.

Proof. The optimal solution to LPLAGRANGE(λ^*) finds the policy $\mathcal{P}_i(t_i)$ for every arm i with $h_i>0$. Therefore, by Lemma 3.1, we must have $\Pi_i(b,t_i)\geq \lambda^*$, and $\Pi_i(b,t)<\lambda^*$ for all $t< t_i$. Furthermore, since the variable x^i_{bt} in the optimal solution to LPLAGRANGE(λ^*) is first non-zero at $t=t_i$, this implies the constraint $\lambda^*+th_i\geq v_{it}p_i$ is first tight at $t=t_i$ by complementary slackness (Lemma 2.4). Further, if this constraint is tight at $t=t_i$, since v_{it} is monotonically increasing, the constraint is feasible for all $t\geq t_i$ only if $h_i>0$. Finally, $\Pi_i(g,1)=r_i(1-\beta_i)\geq \Pi_i(b,t_i)\geq \lambda^*$ follows from Lemma 3.1.

Suppose now that $\lambda^* + th_i \ge v_{it}p_i$ is not tight for any $t \ge 1$. Then, by complementary slackness, we have $x^i_{bt} = 0$ for all $t \ge 1$, which implies $x^i_{gt} = 0$ for all $t \ge 1$. Therefore, the policy $L_i(\lambda^*)$ never plays arm i. This implies $\Pi_i(b,t) \le \lambda^*$ for all $t \ge 1$. Since the excess reward of $L_i(\lambda^*)$ is zero, we have $h_i = 0$. (This can also be shown by complementary slackness.)

3.3.1 Types of Arms

We next classify the arms as follows. In Claim 3.3, let the arms satisfying the first condition $(h_i > 0)$ of the Claim be denoted Type (1), and the remaining arms satisfying $h_i = 0$ be denoted Type (2). Note that type (1) arms are precisely the set S of arms in Fig. 4, so the BALANCEDINDEX policy only plays type (1) arms.

Type (1): Arms in S. We consider the behavior of THRESHOLD-WHITTLE(λ^*) restricted to just these arms. Since $\Pi_i(b,t)$ is monotonically increasing in t, by Claim 3.3, we have the following for the policy of Fig. 4: If the arm is *ready*, the Whittle index is at least λ^* ; if the arm is *bad*, the index is at most λ^* ; and finally, if the arm is *good*, then the modified Whittle index is infinity.

Therefore, THRESHOLD-WHITTLE(λ^*) confined to these arms gives priority to *good* over *ready* over *bad* arms. The only difference with the policy in Fig. 4 is that instead of idling when all arms are *bad*, the policy THRESHOLD-WHITTLE(λ^*) will play some *bad* arm. We now show that this is better than idling.

Claim 3.4. THRESHOLD-WHITTLE(λ^*) executed just over Type (1) arms yields reward at least OPT/2.

Proof. Consider the alternate analysis presented in Section 2.3.3. The INDEX policy from Fig. 4 does not play an arm i in bad state, and achieves change in potential $\Delta\Phi$ of exactly h_i . All we need to show is that if the arm is played instead, the expected change in potential is still at least h_i . The rest of the proof is the same as that of Lemma 2.11. Suppose the arm is played after $t \geq 1$ steps. The expected change in potential is: $\mathbf{E}[\Delta\Phi_t] = v_{it}p_i - h_i(t-1)$. We further have by definition of t_i that $\lambda + t_ih_i = v_{it_i}p_i$. We therefore have $p_iv_{it_i} \geq t_ih_i$. Since v_{it} is a concave function of t with $v_{i0} = 0$, the above implies that for every $t \leq t_i$, we must have $p_iv_{it} \geq th_i$. Therefore, $\mathbf{E}[\Delta\Phi_t] = v_{it}p_i - h_i(t-1) \geq th_i - h_i(t-1) = h_i$.

Type (2): Arms not in S. The only catch now is that THRESHOLD-WHITTLE(λ^*) can sometimes play a type (2) arm whose $h_i = 0$. For such arms, we count their reward and ignore the change in potential.

Lemma 3.5. In Threshold-Whittle(λ^*), if a type (2) arm j preempts the play of a type (1) arm i, either the reward from the former is at least λ^* or the increase in potential of the later $\Delta\Phi$ is at least h_i .

Proof. Suppose that for type (2) arm j, $\Pi_j(g,1) = r_j(1-\beta_j) \ge \lambda^*$. Denote such a state (g,1) as *nice*, and a *nice* type (2) arm has modified index of ∞ . When j was last observed to be good, the arm can be played continuously even if type (1) arms become *ready*. However, for every time step such an event happens, the current reward of playing this *nice* type (2) arm is precisely $r_j(1-\beta_j)$, which is at least λ^* , and the type (1) arms only get better from waiting. When the type (2) arm j was last observed to be bad, preemption can only happen if all type (1) arms are bad, since the Whittle's index of a type (2) arm $\Pi_j(b,\infty) < \lambda^*$. But in this case the increase in potential of arm i is h_i .

Finally, if $\Pi_j(g,1) < \lambda^*$, then by Lemma 3.1, we have $\Pi_j(s,t) < \lambda^*$ for all s = b, g and $t \ge 1$. This implies that such an arm in any state can only preempt type (1) arms that are bad; in that case, the potential Φ of the latter rises by h_i by idling. This completes the proof.

3.3.2 Completing the Proof of Theorem 3.2

To complete the analysis, there are two cases: First, if a *nice* type (2) arm, or a *ready* or *good* type (1) arm is played, then the above discussion implies that the reward plus change in potential ($\Delta\Phi$) of this arm is at least $\lambda^* \geq OPT/2$. In the other case, all type (1) arms are *bad*, and focusing on just these arms, each yields increase in potential for each arm is at least h_i , so that the total reward plus change in potential of these system is at least $\sum_i h_i \geq OPT/2$. This completes the proof, and shows that THRESHOLD-WHITTLE(λ^*) is a 2 approximation. We note that the above analysis extends easily to the variant where $M \geq 1$ arms are simultaneously played per step.

4 The General Technique: MONOTONE Bandits

In this section, we present a general and non-trivial sub-class of restless bandits for which a generalization of the above balancing technique yields a 2-approximate index policy. We term this class MONOTONE bandits, and this captures both the stochastic MAB, as well as the FEEDBACK MAB as special cases.

In MONOTONE bandits, there are n bandit arms. Each arm i can be in one of K states denoted $S_i = \{\sigma_1^i, \sigma_2^i, \ldots, \sigma_K^i\}$. When the arm is not played, its state remains the same and it does not fetch reward. Suppose the arm is in state σ_k^i and is played next after $t \geq 1$ steps. Then, it gains reward $r_k^i \geq 0$, and transitions to one of the states $\sigma_j^i \neq \sigma_k^i$ w.p. $g^i(k,j,t)$, and with the remaining probability stays in state σ_k^i . (For notational convenience, we denote σ_k^i simply as k; the arm it refers to will be clear from the context.) The transition probabilities for different arms are independent. At most one arm is played per step. The goal is to find a policy for playing the arms so that the infinite horizon time-average reward is maximized.

In addition, we have the following key properties about the transition probabilities:

Separability Property: We assume that $g^i(k,j,t)$ is of the form $f^i_k(t)q^i(k,j)$. The function $f^i_k(t) \in [0,1]$ for positive integers t can be thought of as an "escape probability" from the state $\sigma_k \in \mathcal{S}_i$. Conditioned of the escape, the state changes to $\sigma_j \in \mathcal{S}_i$ with probability $q^i(k,j)$, thus $\sum_{j \neq k} q^i(k,j) \leq 1$.

Monotone Property: For every arm i and state $k \in S_i$, we have: $f_k^i(t) \leq f_k^i(t+1)$ for every t.

The above properties are necessary in some sense: We show in Section 4.5 that when the monotone property is relaxed, the problem becomes n^{ϵ} -hard to approximate. Further, if the separability property is not satisfied, then Whittle's LP on which the analysis of this section is based, has $\Omega(n)$ gap.

Motivation and Special Cases. Intuitively, MONOTONE bandit models optimization scenarios in which uncertainty increases: when an arm is just played and we observe its state, we are most certain that our

observation still holds true the next time step. However, the non-decreasing nature of f implies that as time goes on, the "escape probability" increases and the previous observation becomes less and less reliable. This serves as a model for certain POMDPs, such as the FEEDBACK MAB.

Observe that the MONOTONE bandits generalizes the FEEDBACK MAB. For the states $S_i = \{g, b\}$, set $q^i(g,b) = q^i(b,g) = 1$ and $f_g^i(t) = 1 - u_{it}$ and $f_b^i(t) = v_{it}$. Recall from Fact 2.2 that u_{it}, v_{it} are respectively the probabilities of observing the state g when the state last observed t steps ago was g and g, and that $1 - u_{it}, v_{it}$ are both monotonically increasing. We also note that MONOTONE bandits generalizes the stochastic MAB by setting $f_k^i(t) = 1$ for all t.

4.1 High Level Idea

Unlike the FEEDBACK MAB problem, in MONOTONE bandits, there is no longer a clear distinction between "good" and "bad" states. Note however that an equivalent way of finding λ such that $\lambda = \sum_{i=1}^n H_i(\lambda)$ is to treat λ as a variable and enforce $\lambda = \sum_{i=1}^n h_i$ as a constraint in the dual of Whittle's LP. By taking this approach, the variables p_k^i (now one for each state $k \in \mathcal{S}_i$) can be interpreted as dual potentials, and the dual constraints are in terms of the expected potential change of playing in state $k \in \mathcal{S}_i$. Based on the sign of this potential change, we can classify the states into "good" and "bad" via complementary slackness. Our index policy continuously exploits arms in "good" states, and waits until the dual constraint goes tight (i.e., the arm becomes "ready") before playing in "bad" states. We formalize the previous potential-based argument using a Lyapunov function and show a 2-approximation. We note that the LP-duality approach is entirely equivalent to the Lagrangean approach; however, it leads to a different interpretation of variables which is more generalizable.

Technical Assumptions. For simplicity of the exposition, we assume the monotone functions in this section are piece-wise linear with poly-size specification – see definition 5 for a formal definition. As shown in the previous section, these results do extend to a wider class of differentiable functions, such as those in FEEDBACK MAB.

We also assume that for each arm i, the graph, where the vertices are $k \in S_i$ and a directed edge (j, k) exists if $q^i(j, k) > 0$, is strongly connected. Since we consider the infinite horizon time average reward, assume that the policy is ergodic and can choose the start state of each arm. These assumptions do not simplify the problem, as it remains NP-HARD (see Section 4.5).

4.2 Whittle's LP and its Dual

As with FEEDBACK MAB, for each arm i and $k \in \mathcal{S}_i$, we have variables $\{x_{kt}^i, t \geq 1\}$. These variables capture the probabilities (in the execution of the optimal policy) of the event: Arm i is in state k, was last played t steps ago, and is played at the current step. These quantities are well-defined for ergodic policies. Whittle's LP is presented in Figure 7. Let its optimal value be denoted OPT. The LP effectively encodes the constraints on the evolution of the state of each arm separately, connecting them only by the constraint that at most one arm is played in expectation every step. The first constraint simply states that the at any step, at most one arm is played; the second constraint encodes that each arm can be in at most one possible state at any time step; and the final constraint encodes that the rate of entering state $k \in \mathcal{S}_i$ is the same as the rate of exiting this state. This LP will clearly be a relaxation of the optimal policy; the details are the same as the proof of Lemma 2.3.

This LP has infinite size, and we will fix that aspect in this section. In particular, we now show that the LP has polynomial size when the f_k^i are piece-wise linear with poly-size specification.

Definition 5. Given $i, k \in S_i$, $f_k^i(t)$ is specified as the piece-wise linear function that passes through breakpoints $(t_1 = 1, f_k^i(1)), (t_2, f_k^i(t_2)), \ldots, (t_m, f_k^i(t_m))$. Denote the set $\{t_1, t_2, \ldots, t_m\}$ as \mathcal{W}_k^i . Therefore, for

$$\begin{aligned} & \text{Maximize} \quad \sum_{i=1}^{n} \sum_{k \in \mathcal{S}_{i}} \sum_{t \geq 1} r_{k}^{i} x_{kt}^{i} & \text{(Whittle)} \\ & \sum_{i=1}^{n} \sum_{k \in \mathcal{S}_{i}} \sum_{t \geq 1} x_{kt}^{i} & \leq \quad 1 \\ & \sum_{k \in \mathcal{S}_{i}} \sum_{t \geq 1} t x_{kt}^{i} & \leq \quad 1 \\ & \sum_{j \in \mathcal{S}_{i}, j \neq k} \sum_{t \geq 1} x_{kt}^{i} q^{i}(k, j) f_{k}^{i}(t) & = \quad \sum_{j \in \mathcal{S}_{i}, j \neq k} \sum_{t \geq 1} x_{jt}^{i} q^{i}(j, k) f_{j}^{i}(t) & \forall k \in \mathcal{S}_{i} \\ & x_{kt}^{i} & \geq \quad 0 & \forall i, k, t \end{aligned}$$

Figure 7: The linear program (WHITTLE).

two consecutive points $t_1, t_2 \in \mathcal{W}_k^i$ with $t_1 < t_2$, the function f_k^i is specified at t_1 and t_2 . For $t \in (t_1, t_2)$, we have $f_k^i(t) = ((t_2 - t)f_k^i(t_1) + (t - t_1)f_k^i(t_2))/(t_2 - t_1)$. For $t \ge t_m$, we have $f_k^i(t) = f_k^i(t_m)$. We assume that \mathcal{W}_k^i has poly-size specification.

Consider the dual of the above relaxation. The first constraint has multiplier λ , the second set of constraints have multipliers h_i , and the final equality constraints have multipliers p_k^i . For notational convenience, define:

$$\Delta P_k^i = \sum_{j \in \mathcal{S}_i, j \neq k} \left(q^i(k, j) (p_j^i - p_k^i) \right) \tag{2}$$

Note that ΔP_k^i is a variable that depends on the dual variables p_*^i . We obtain the following dual.

$$\begin{aligned} & \text{Minimize} \quad \lambda + \sum_{i=1}^n h_i \\ & \lambda + t h_i & \geq \quad r_k^i + f_k^i(t) \Delta P_k^i \quad \forall i,k \in \mathcal{S}_i, t \geq 1 \\ & \lambda, h_i & \geq \quad 0 & \forall i \end{aligned}$$

Since $f_k^i(t)$ is piece-wise linear, for two consecutive break-points $t_1 < t_2$ in \mathcal{W}_k^i , the constraint $\lambda + th_i \ge r_k^i + f_k^i(t)\Delta P_k^i$ is true for all $t \in [t_1, t_2]$ iff it is true at t_1 and at t_2 . This means that the constraints for $t \notin \mathcal{W}_k^i$ are redundant. Therefore, the above dual is equivalent to the one presented in Figure 8, which we denote (WHITTLE-DUAL).

$$\begin{array}{lll} \text{Minimize} & \lambda + \sum_{i=1}^n h_i & (\text{Whittle-Dual}) \\ & \lambda + th_i & \geq & r_k^i + f_k^i(t) \Delta P_k^i & \forall i,k \in \mathcal{S}_i, t \in \mathcal{W}_k^i \\ & \lambda, h_i & \geq & 0 & \forall i \end{array}$$

Figure 8: The polynomial size dual of Whittle's LP, which we denote (WHITTLE-DUAL).

Taking the dual of the above program, we finally obtain a polynomial size relaxation for MONOTONE bandits. Since this poly-size LP only differs from (WHITTLE) in restricting t to lie in the relevant set \mathcal{W}_k^i , and since it will not be explicitly needed in the remaining discussion, we omit writing it explicitly.

4.3 The Balanced Linear Program

We do not solve Whittle's relaxation. Instead, we solve the modification of (WHITTLE-DUAL) from Figure 8, which we denote (BALANCE). This is shown in Figure 9. The additional constraint in (BALANCE) (as in FEEDBACK MAB) is the constraint $\lambda = \sum_{i=1}^{n} h_i$.

$$\begin{array}{lll} \text{Minimize} & \lambda + \sum_{i=1}^n h_i & \text{(Balance)} \\ \\ \lambda + th_i & \geq & r_k^i + f_k^i(t) \Delta P_k^i & \forall i,k \in \mathcal{S}_i, t \in \mathcal{W}_k^i \\ \\ \lambda & = & \sum_{i=1}^n h_i \\ \\ \lambda, h_i & \geq & 0 & \forall i \end{array}$$

Figure 9: The dual linear program (BALANCE) for MONOTONE MAB.

The primal linear program corresponding to (BALANCE) is the following (where we place an unconstrained multiplier ω to the final constraint of (BALANCE)):

$$\begin{array}{llll} \text{Maximize} & \sum_{i=1}^n \sum_{k \in \mathcal{S}_i} \sum_{t \in \mathcal{W}_k^i} x_{kt}^i & \text{(Primal-Balance)} \\ & \sum_{i=1}^n \sum_{k \in \mathcal{S}_i} \sum_{t \in \mathcal{W}_k^i} x_{kt}^i & \leq & 1-\omega \\ & \sum_{k \in \mathcal{S}_i} \sum_{t \in \mathcal{W}_k^i} t x_{kt}^i & \leq & 1+\omega & \forall i \\ \sum_{j \neq k} \sum_{t \in \mathcal{W}_k^i} x_{kt}^i q^i(k,j) f_k^i(t) & = & \sum_{j \neq k} \sum_{t \in \mathcal{W}_j^i} x_{jt}^i q^i(j,k) f_j^i(t) & \forall i,k \\ & x_{kt}^i & \geq & 0 & \forall i,k,t \end{array}$$

The first step of the algorithm is to solve the linear program (BALANCE). Clearly the value of this LP is at least OPT. We now show the following properties of the optimal solution to (BALANCE) using complementary slackness conditions between (BALANCE) and (PRIMAL-BALANCE). From now on, we only deal with the optimal solutions to the above programs, so all variables correspond to the optimal setting.

Lemma 4.1. Recall that OPT is the optimal value to (WHITTLE). Since any feasible solution to (BALANCE) is feasible to (WHITTLE-DUAL), in the optimal solution to (BALANCE), $\lambda = \sum_{i=1}^{n} h_i \ge OPT/2$.

The next lemma is the crux of the analysis, where for any arm being played in any state, we use complementary slackness to explicitly relate the dual variables to the reward obtained. Note that unlike the analyses of primal-dual algorithms, our proof needs to use both the *exact* primal as well as dual complementary slackness conditions. This aspect requires us to actually solve the dual optimally.

Lemma 4.2. One of the following is true for the optimal solution to (BALANCE): Either there is a trivial 2-approximation by repeatedly playing the same arm; or for every arm i with $h_i > 0$ and for every state $k \in \mathcal{S}_i$, there exists $t \in \mathcal{W}_k^i$ such that the following LP constraint is tight with equality.

$$\lambda + th_i \ge r_k^i + f_k^i(t)\Delta P_k^i \tag{3}$$

Proof. Note that if $\omega \le -1$ or $\omega \ge 1$, then the values of (PRIMAL-BALANCE) is 0, but the optimal value of (PRIMAL-BALANCE) is at least OPT > 0. Thus, in the optimal solution to (PRIMAL-BALANCE), $\omega \in (-1,1)$.

The optimal solutions to (BALANCE) and (PRIMAL-BALANCE) satisfy the following complementary slackness conditions (recall from above that $\omega > -1$ so that $1 + \omega > 0$):

$$h_i > 0$$
 \Rightarrow $\sum_{k \in \mathcal{S}_i} \sum_{t \in \mathcal{W}_k^i} t x_{kt}^i = 1 + \omega > 0$ (4)

$$\lambda + th_i > r_k^i + f_k^i(t)\Delta P_k^i \qquad \Rightarrow \qquad x_{kt}^i = 0 \tag{5}$$

Suppose that for some i such that $h_i > 0$, and for some $k \in \mathcal{S}_i$, we have $\lambda + th_i > r_k^i + f_k^i(t)\Delta P_k^i$ for every $t \in \mathcal{W}_k^i$. By condition (5), $x_{kt}^i = 0 \ \forall t \in \mathcal{W}_k^i$, which trivially implies that $x_{kt}^i f_k^i(t) = 0 \ \forall t \in \mathcal{W}_k^i$.

Now, given that for a certain arm i and state k, $x_{kt}^i f_k^i(t) = 0 \ \forall t$. Therefore, in the following constraint in (PRIMAL-BALANCE):

$$\sum_{l \neq k} \sum_{t \in \mathcal{W}_k^i} x_{kt}^i f_k^i(t) q^i(k, l) = \sum_{l \neq k} \sum_{t \in \mathcal{W}_l^i} x_{lt}^i f_l^i(t) q^i(l, k)$$

the LHS is zero because $x_{kt}^i f_k^i(t) = 0$, which means the RHS is zero. Since all variables are non-negative, this implies that for any $j \in \mathcal{S}_i$ with $q^i(j,k) > 0$, we have $x_{jt}^i f_j^i(t) = 0$ for all $t \in \mathcal{W}_j^i$.

Recall (from Section 4) that we assumed the graph on the states with edges from j to k if $q^i(j,k) > 0$ is strongly connected. Therefore, by repeating the above argument, we get $\forall j,t \in \mathcal{W}^i_j, x^i_{jt}f^i_j(t) = 0$.

By Condition (4), since $h_i>0$, there exists $j\in\mathcal{S}_i$ and $t\in\mathcal{W}^i_j$, such that $x^i_{jt}>0$ (or else the sum in Condition (4) is zero). By what we proved in the previous paragraph, this implies that $f^i_j(t)=0$, which implies that $f^i_j(1)=0$ by the MONOTONE property. Since $x^i_{jt}>0$, using Condition (5) and plugging in $f^i_j(t)=0$, we get $\lambda+th_i=r^i_j$. Moreover, by plugging in $f^i_j(1)=0$ into the t=1 constraint of (BALANCE), we get $\lambda+h_i\geq r^i_j$. These two facts imply that $\lambda+h_i=r^i_j$. The above implies that the policy that starts with arm i in state j and always plays this arm obtains per-step reward $\lambda+h_i>OPT/2$.

In the remaining discussion, we assume that the above lemma does not find an arm i that yields reward at least OPT/2. This means that $\forall i, k$, there exists some $t \in \mathcal{W}_k^i$ that makes Inequality (3) tight.

Lemma 4.3. For any arm i such that $h_i > 0$, and state $k \in S_i$, if $\Delta P_k^i < 0$, then:

$$\lambda + h_i = r_k^i + f_k^i(1)\Delta P_k^i$$

Proof. By Lemma 4.2 and our assumption above, Inequality (3) in Lemma 4.2 is tight for some $t \in \mathcal{W}_k^i$. If it is not tight for t = 1, then since $f_k^i(t)$ is non-decreasing in t and since $\Delta P_k^i < 0$, it will not be tight for any t. Thus, we have a contradiction.

4.4 The BALANCEDINDEX Policy

Start with the optimal solution to (BALANCE). First throw away the arms for which $h_i = 0$. By Lemma 4.1, for the remaining arms, $\sum_i h_i \ge OPT/2$. Define the following quantities for each of these arms.

Definition 6. For each i ($h_i > 0$ by assumption) and state $k \in \mathcal{S}_i$, let t_k^i be the smallest value of $t \in \mathcal{W}_k^i$ for which $\lambda + th_i = r_k^i + f_k^i(t)\Delta P_k^i$ in the optimal solution to (BALANCE). By Lemma 4.2, t_k^i is well-defined for every $k \in \mathcal{S}_i$.

Definition 7. For arm i, partition the states S_i into states G_i , I_i as follows:

1.
$$k \in \mathcal{G}_i$$
 if $\Delta P_k^i < 0$. (By Lemma 4.3, $t_k^i = 1$.)

2.
$$k \in \mathcal{I}_i$$
 if $\Delta P_k^i \geq 0$.

With the notation above, the policy is now presented in Figure 10. In this policy, if arm i has been in state $k \in \mathcal{I}_i$ for less than t_k^i steps, it is defined to be "not ready" for play. Once it has waited for t_k^i steps, it becomes "ready" and can be played. Moreover, if arm i moves to a state in $k \in \mathcal{G}_i$, it is continuously played until it moves to a state in \mathcal{I}_i .

Intuitively, the states in \mathcal{G}_i are the "exploitation" or "good" states. On the contrary, the states in \mathcal{I}_i are "exploration" or "bad" states, so the policy waits until it has a high enough probability of exiting these states

BALANCEDINDEX Policy

- 1. **Exploit:** Some arm i moves to a state $k \in \mathcal{G}_i$:
 - (a) Play this arm exclusively as long as it remains in a state in G_i .
 - (b) **Goto** step (2).
- 2. Explore:
 - (a) Play any "ready" arm i in state $k \in \mathcal{I}_i$. (If no arm is "ready", do not play at this step.)
 - (b) If the arm moves to state in G_i : goto Step (1), else goto Step (2a).

Figure 10: The BALANCEDINDEX Policy for MONOTONE MAB.

before playing them. In both cases, t_k corresponds to the "recovery time" of the state, which is 1 in a "good" state but could be large in a "bad" state.

Lyapunov Function Analysis. We use a Lyapunov (potential) function argument to show that the policy described in Figure 10 is a 2-approximation. Define the potential Φ_i for each arm i at any time as follows. (Recall the definition of t_k^i from Definition 6, as well as the quantities λ , h_i from the optimal solution of (BALANCE).)

Definition 8. If arm i moved to state $k \in S_i$ some y steps ago ($y \ge 1$ by definition), the potential Φ_i is $p_k^i + h_i(\min(y, t_k^i) - 1)$.

Therefore, whenever the arm i enters state k, its potential is p_k^i . If $k \in \mathcal{I}_i$, the potential then increases at rate h_i for $t_k^i - 1$ steps, after which it remains fixed until the arm is played. Our policy plays arm i only if its current potential is $p_k^i + h_i(t_k^i - 1)$.

We finally complete the analysis in the following lemma. The proof crucially uses the "balance" property of the dual, which implies that $\lambda = \sum_i h_i \geq OPT/2$. Let Φ_T denote the total potential, $\sum_{i=1}^n \Phi_i$, at any step T and let R_T denote the total reward accrued until that step. Define the function $\mathcal{L}_T = t \cdot OPT/2 - R_T - \Phi_T$. Let $\Delta R_T = R_{T+1} - R_T$ and $\Delta \Phi_T = \Phi_{T+1} - \Phi_T$.

Lemma 4.4. \mathcal{L}_T is a Lyapunov function. i.e., $\mathbf{E}[\mathcal{L}_{T+1}|\mathcal{L}_T] \leq \mathcal{L}_T$. Equivalently, at any step:

$$\mathbf{E}[\Delta R_T + \Delta \Phi_T | R_T, \Phi_T] \ge OPT/2$$

Proof. At a given step, suppose the policy does nothing, then all arms are "not ready". The total increase in potential is precisely $\Delta \Phi_T = \sum_i h_i \ge OPT/2$.

On the other hand, suppose that the policy plays arm i, which is currently in state k and has been in that state for $y \ge t_k^i$ steps. The change in reward $\Delta R_T = r_k^i$. Moreover, the current potential of the arm must be $\Phi_T = p_k + h_i(t_k^i - 1)$. The new potential follows the following distribution:

$$\Phi_{T+1} = \begin{cases} p_j^i, & \text{with probability } f_k^i(y)q^i(k,j) \ \forall j \neq k \\ p_k^i, & \text{with probability } 1 - \sum_{j \neq k} f_k^i(y)q^i(k,j) \end{cases}$$

Therefore, if arm i is played, the change in potential is:

$$\mathbf{E}[\Delta\Phi_T] = f_k^i(y) \sum_{j \in \mathcal{S}_i, j \neq k} \left(q^i(k, j) (p_j^i - p_k^i) \right) - h_i(t_k^i - 1)$$

From the description of the INDEX policy, $y=t_k^i=1$ if $k\in\mathcal{G}_i$. Therefore, y might be strictly greater than t_k^i only when $k\in\mathcal{I}_i$. In that case $\Delta P_k^i\geq 0$ by Definition 7, so that $f_j^i(y)\Delta P_k^i\geq f_j^i(t_k^i)\Delta P_k^i$ by the MONOTONE property (since $y\geq t_k^i$).

Therefore, for the arm i being played, regardless of whether $k \in \mathcal{G}_i$ or $k \in \mathcal{I}_i$,

$$\Delta R_T + \mathbf{E}[\Delta \Phi_T] = r_k^i + f_k^i(y) \Delta P_k^i - h_i (t_k^i - 1)$$

$$\geq r_k^i + f_k^i (t_k^i) \Delta P_k^i - h_i t_k^i + h_i$$

$$= \lambda + h_i > OPT/2$$

where the last equality follows from the definition of t_k^i (Definition 6). Since the potentials of the arms not being played do not decrease (since all $h_l > 0$), the total change in reward plus potential is at least OPT/2. This completes the proof. Refer Figure 11 for a "picture proof" when $k \in \mathcal{I}_i$.

By their definition, the potentials Φ_T are bounded independent of the time horizon, by telescoping summation, the above lemma implies that $\lim_{T\to\infty}\frac{\mathbf{E}[R_T]}{T}\geq OPT/2$. We finally have:

Theorem 4.5. The BALANCEDINDEX policy is a 2 approximation for MONOTONE bandits.

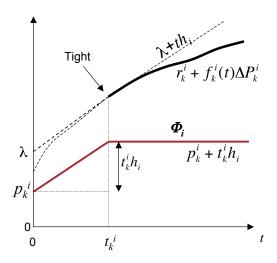


Figure 11: Proof of Lemma 4.4. The growth of the potential Φ is shown on the lower piece-wise linear function. The upper set of curves represent the LHS and RHS of the LP constraints for state $k \in \mathcal{S}_i$. The tight point t_k^i is where the potential switches to being constant.

4.5 Lower Bounds: Necessity of Monotonicity and Separability

We show that MONOTONE bandits is NP-Hard, and that if the MONOTONE property is relaxed even slightly, the problem either has $\Omega(n)$ integrality gap for Whittle's LP, or becomes n^{ϵ} -hard to approximate.

Input Specification. In the above discussion, we assumed the input to the MONOTONE bandits problem is specified by polynomial size state spaces S_i for each arm; the associated matrices q^i , and functions $f_k^i(t)$ that are piecewise linear with poly-size specification. We can model this problem as a restless bandit problem in the sense defined in literature by replacing each state $k \in S_i$ with exponentially many states $\{k_t, t \in \mathbf{Z}^+\}$; if the arm is not played, it transitions deterministically from state k_t to k_{t+1} , but if played in state k_t , it transitions w.p. $q^i(k,j)f_k^i(t)$ to state j_1 for each $j \in S_i$, and with the remaining probability transitions to k_1 . The reduction uses exponentially many states, and is unlike the typical formulation of restless bandits that assumes the state space of each arm is poly-bounded. (The PSPACE-Hardness proofs of restless bandits [41]

assumes poly-bounded state space as well.) We therefore need to use different NP-Hardness proofs for our compact input specifications.

Theorem 4.6. For the special case of the problem with K = 2 states per arm and n arms, the following are true even when the functions f_k^i are piece-wise linear with poly-size specification:

- 1. Computing the optimal ergodic policy for MONOTONE bandits is NP-Hard.
- 2. If the MONOTONE property is relaxed to allow arbitrary (possibly non-monotone) functions f, then the problem becomes n^{ϵ} hard to approximate unless P = NP.

Proof. We reduce from the following periodic scheduling problem, which is shown to be NP-Complete in [9]: Given n positive integers l_1, l_2, \ldots, l_n such that $\sum_{i=1}^n 1/l_i \leq 1$, is there an infinite sequence of integers $\{1, 2, \ldots, n\}$ such that for every $i \in \{1, 2, \ldots, n\}$, all consecutive occurrences of i are exactly l_i elements apart. Given an instance of this problem, for each $i \in \{1, 2, \ldots, n\}$, we define an arm i with a "good" state g and a "bad" state w.

For part 1, for every arm i, let $r_g^i=1$, and $r_w^i=0$. Set $q^i(g,w)=1$ and $f_g^i(t)=1$ for all t. Moreover, set $q^i(w,g_i)=1$ and $f_w^i(t)=0$ if $t\leq 2l_i-2$ and 1 otherwise. Suppose for a moment that we only have arm i, then the optimal policy will play the arm exactly $2l_i-1$ steps after it is observed to be in w, and the arm will transition to state g. The policy will then play the arm in state g to obtain reward 1, and the arm will transition back to state w. Since this policy is periodic with period $2l_i$, it yields long term average reward exactly $\frac{1}{2l_i}$. It is easy to see that any other ergodic policy of playing this arm yields strictly smaller reward per step. Any policy of playing all the arms therefore has total reward of at most $\sum_{i=1}^n \frac{1}{2l_i}$. But for any ergodic policy, the reward of $\sum_{i=1}^n \frac{1}{2l_i}$ is achievable only if each arm i is played according to its individual optimal policy, which is twice in succession every $2l_i$ steps. But deciding whether this is possible is equivalent to solving the periodic scheduling problem on the l_i . Therefore, deciding whether the optimal policy to the MONOTONE bandit problem yields reward $\sum_{i=1}^n \frac{1}{2l_i}$ is NP-Hard.

For part 2, we make w a trapping state with no reward. For arm i, set $q^i(g,w)=q^i(w,g)=1$; and $f^i_g(l_i)=0$ and $f^i_g(t)=1$ for all $t\neq l_i$. Furthermore, $f^i_w(t)=0$ for all t. Also set $r^i_g=l_i$ and $r^i_w=0$. Therefore, for any arm i, any policy will obtain reward from this arm if and only if it chooses the start state to be g, and plays the arm periodically once every l_i steps to obtain average reward 1. Therefore, approximating the value of the optimal policy is the same as approximating the size of the largest subset of $\{l_1, l_2, \ldots, l_n\}$ so that this subset induces a periodic schedule. The NP-Hardness proof of periodic scheduling in [9] shows that this problem as hard as approximating the size of the largest subset of vertices in a graph whose induced subgraph is bipartite, which is n^ϵ hard to approximate [38] unless P=NP.

In the above proof, we showed that the problem becomes hard to approximate if the transition probabilities are non-monotone. However, that does not address the question of how far we can push our technique. We give a negative result by showing that Whittle's LP can have arbitrarily large gap even if the MONOTONE bandit problem is slightly generalized by preserving the monotone nature of the transition probabilities, but removing the additional *separable* structure that they should be of the form $f_k^i(t)q^i(k,j)$. In other words, the transition probability from state k to state $j \neq k$ if played after t steps is $q_{kj}^i(t)$ – these are arbitrary monotonically non-decreasing functions of t. We insist $\sum_{j\neq k}q_{jk}^i(t)\leq 1$ for all k,t to ensure feasibility. We show that Whittle's LP has $\Omega(n)$ gap for this generalization.

Theorem 4.7. If the separability assumption on transition probabilities is relaxed, Whittle's LP has $\Omega(n)$ gap even with K=3 states per arm.

Proof. The arms are all identical. Each has 3 states, $\{g,b,a\}$. State a is an absorbing state with 0 reward. State g has reward 1, and state b has reward 0. The transition probabilities are as follows: $q_{ab}(t) = q_{ag}(t) = q_{ag}(t)$

0. Further, $q_{gb}(t) = 1/2$, $q_{ga}(1) = 0$; and $q_{ga}(t) = 1/2$ for $t \ge 2$. Finally, $q_{ba}(t) = q_{bg}(t) = 0$ for t < 2n - 1; $q_{bg}(2n - 1) = 1/2$; $q_{ba}(2n - 1) = 0$; and $q_{ba}(t) = q_{bg}(t) = 1/2$ for $t \ge 2n$.

A feasible single arm policy involves playing the arm in state b after exactly 2n-1 steps (w.p. 1/2, the state transitions to g), and continuously in state g (w.p. 1/2, the state transitions to g). This policy never enters state g. The average rate of play is g0. The per-step reward of this policy is g0. Whittle's LP chooses this policy for each arm so that the total rate of play is g1 and the objective is g2.

Now consider any feasible policy that plays at least 2 arms. If one of these arms is in state g, there is a non-zero probability that either this arm is played after t > 1 steps, or the other arm in state b is played after $t \ge 2n$ steps. In either case, w.p. 1/2, the arm enters absorbing state. Since this is an infinite horizon problem, the above event happens w.p. 1. Therefore, any feasible policy is restricted to playing only one arm in the long run, and obtains reward at most 1/n.

5 MONOTONE Bandits: Multiple Simultaneous Plays of Varying Duration

In this section, we extend the index policy for MONOTONE bandits to handle multiple plays of varying duration. We use the same problem description as in Section 4, except we assume there are $M \ge 1$ players, each of which can play one arm every time step. (Therefore, M plays can proceed simultaneously per step.)

Furthermore, we assume that if arm i in state $k \in \mathcal{S}_i$ is played, this play takes $L_k^i \geq 1$ steps and during this time, this player cannot play another arm. We note that the values L_k^i are fixed beforehand, and the players are aware of these values. When the player plays arm i in state k, he/she is forced to remain on arm i for L_k^i steps, and he/she only receives one reward of magnitude r_k^i , at the beginning of this "blocking" period.

Suppose when the current play begins, the previous play had ended $t \geq 1$ steps ago. Then, at the end of the current play, the arm transitions to one of the states $j \neq k$ w.p. $q^i(k,j)f^i_k(t)$, and with the remaining probability stays in state k. In Section 4, we focused on the case where M=1 and all $L^i_k=1$.

Since the overall algorithm and analysis are very similar to that in Section 4, we simply outline the differences. First, Whittle's LP gets modified as follows:

$$\begin{array}{ccccc} \sum_{i=1}^{n} \sum_{k \in \mathcal{S}_{i}} \sum_{t \geq 1} L_{k}^{i} x_{kt}^{i} & \leq & M \\ \sum_{k \in \mathcal{S}_{i}} \sum_{t \geq 1} (t + L_{k}^{i} - 1) x_{kt}^{i} & \leq & 1 & \forall i \\ \sum_{j \in \mathcal{S}_{i}, j \neq k} \sum_{t \geq 1} x_{kt}^{i} q^{i}(k, j) f_{k}^{i}(t) & = & \sum_{j \in \mathcal{S}_{i}, j \neq k} \sum_{t \geq 1} x_{jt}^{i} q^{i}(j, k) f_{j}^{i}(t) & \forall k \in \mathcal{S}_{i} \\ x_{kt}^{i} & \geq & 0 & \forall i, k \in \mathcal{S}_{i}, t \geq 1 \end{array}$$

In the above formulation, the first constraint merely encodes that in expectation M arms are played per step. Note that each play of arm i in state k lasts L_k^i steps, and the play begins with probability x_{kt}^i , so that the steady state probability that arm i in state k is being played at any time step is $\sum_{t\geq 1} L_k^i x_{kt}^i$. Note now that if the play begins after t steps, then the arm was idle for t-1 steps before this event. Therefore, the quantity $\sum_{t\geq 1} (t+L_k^i-1)x_{kt}^i$ would be the steady state probability that the arm i is in state k. This summed over all k must be at most 1 for any arm i. This is the second constraint. The final constraint encodes that the rate of leaving state k in steady state (LHS) must be the same as the rate of entering state k (RHS).

5.1 Balanced Program and Complementary Slackness

The balanced linear program is in Fig. 12. (Recall the definition of $\Delta P_k^i(t)$ from Equation 2.) Next, Lemma 4.2 gets modified as follows:

$$\begin{array}{lll} \text{Minimize} & M\lambda + \sum_{i=1}^n h_i & \text{(Balance)} \\ L_k^i(\lambda + h_i) + (t-1)h_i & \geq & r_k^i + f_k^i(t)\Delta P_k^i(t) & \forall i,k \in \mathcal{S}_i, t \in \mathcal{W}_k^i \\ & M\lambda & = & \sum_{i=1}^n h_i \\ & \lambda, h_i & \geq & 0 & \forall i \end{array}$$

Figure 12: The linear program (BALANCE) that we actually solve.

Lemma 5.1. In the optimal solution to (BALANCE), one of the following is true for every arm i with $h_i > 0$: Either repeatedly playing the arm yields per-step reward at least $\lambda + h_i$; or for every state $k \in S_i$, there exists $t \in W_k^i$ such that the following LP constraint is tight with equality.

$$L_k^i(\lambda + h_i) + (t - 1)h_i \ge r_k^i + f_k^i(t)\Delta P_k^i(t) \tag{6}$$

We next split the arms into two types:

Definition 9. 1. Arm $i \in U_1$ if repeatedly playing it yields average per-step reward at least $\lambda + h_i$. Our policy described in the next section favors these arms and continuously plays them.

2. Arm $i \in U_2$ if $i \notin U_1$ and $h_i > 0$. Note that for $i \in U_2$, $\forall k, \exists t \in \mathcal{W}_k^i$ that makes Inequality (6) tight.

Lemma 5.2. For any arm $i \in U_2$ and state $k \in S_i$, if $\Delta P_k^i(t) < 0$, then:

$$L_k^i(\lambda + h_i) = r_k^i + f_k^i(1)\Delta P_k^i(t)$$

5.2 BALANCEDINDEX Policy

Definition 10. For each $i \in U_2$ and state $k \in S_i$, let t_k^i be the smallest value of $t \in W_k^i$ for which Inequality (3) is tight. By Lemma 4.2, t_k^i is well-defined for every $k \in S_i$.

Definition 11. For arm $i \in U_2$, partition the states S_i into states G_i , I_i as follows:

- 1. $k \in \mathcal{G}_i$ if $\Delta P_k^i(t) < 0$. (By Lemma 5.2, $t_k^i = 1$.)
- 2. $k \in \mathcal{I}_i$ if $\Delta P_h^i(t) > 0$.

Finally, the BALANCEDINDEX policy is described in Figure 13. Note that any arm $i \in U_2$ that is observed to be in a state in \mathcal{G}_i is continuously played until its state transitions into \mathcal{I}_i . This preserves the invariant that at most $M - |U_1|$ arms $i \in U_2$ are in states $k \in \mathcal{G}_i$ at any time step.

5.3 Lyapunov Function Analysis

Define the potential for each arm in U_2 at any time as follows.

Definition 12. If arm $i \in U_2$ moved to state $k \in S_i$ some y steps ago $(y \ge 1)$ by definition, the potential is $p_k^i + h_i(\min(y, t_k^i) - 1)$.

Therefore, whenever the arm $i \in U_2$ enters state k, its potential is p_k^i . If $k \in \mathcal{I}_i$, the potential then increases at rate h_i for $t_k^i - 1$ steps, after which it remains fixed until a play completes for it. When our policy decides to play arm $i \in U_2$, its current potential is $p_k^i + h_i(t_k^i - 1)$.

We finally complete the analysis in the following lemma. The proof crucially uses the "balance" property of the dual, which states that $M\lambda = \sum_i h_i \geq OPT/2$. Let Φ_T denote the total potential at any step T and let R_T denote the total reward accrued until that step. Define the function $\mathcal{L}_T = T \cdot OPT/2 - R_T - \Phi_T$. Let $\Delta R_T = R_{T+1} - R_T$ and $\Delta \Phi_T = \Phi_{T+1} - \Phi_T$.

BALANCEDINDEX Policy at any time step

Continuously play $\min(M, |U_1|)$ arms in U_1 . /* Execute the remaining policy only if $|U_1| < M$ */

Invariant: At most $M - |U_1|$ arms $i \in U_2$ are in states $k \in \mathcal{G}_i$.

If a player becomes free, prioritize the available arms in U_2 as follows:

- (a) **Exploit:** Choose to play an arm i in state $k \in \mathcal{G}_i$.
- (b) **Explore:** If no "good" arm available, play any "ready" arm i in state $k \in \mathcal{I}_i$.
- (c) Idle: If no "good" or "ready" arm available, then idle.

Figure 13: The INDEX Policy.

Lemma 5.3. \mathcal{L}_T is a Lyapunov function. i.e., $\mathbf{E}[\mathcal{L}_{T+1}|\mathcal{L}_T] \leq \mathcal{L}_T$. Equivalently, at any step:

$$\mathbf{E}[\Delta R_T + \Delta \Phi_T | R_T, \Phi_T] \ge OPT/2$$

Proof. Arms $i \in U_1$ are played continuously and yield average per step reward $\lambda + h_i$, so that for any such arm i being played, $\mathbf{E}[\Delta R_T] = \lambda + h_i$.

Next focus on arms $i \in U_2$. As before, it is easy to show that when played, regardless of whether $k \in \mathcal{G}_i$ or $k \in \mathcal{I}_i$,

$$\Delta R_{i} + \mathbf{E}[\Delta \Phi_{i}] = r_{k}^{i} + f_{k}^{i}(y) \sum_{j \in \mathcal{S}_{i}, j \neq k} \left(q^{i}(k, j) (p_{j}^{i} - p_{k}^{i}) \right) - h_{i}(t_{k}^{i} - 1)$$

$$\geq r_{k}^{i} + f_{k}^{i}(t_{k}^{i}) \Delta P_{k}^{i}(t) - h_{i}(t_{k}^{i} - 1) = L_{k}^{i}(\lambda + h_{i})$$

where the last equality follows from the definition of t_k^i (Definition 10). Since the play lasts L_k^i time steps, the amortized per step change for the duration of the play, $\Delta R_T + \mathbf{E}[\Delta \Phi_T]$, is equal to $\lambda + h_i$.

We finally bound the increase in reward plus potential at any time step. At step T, let S_g denote the arms in U_1 and those in U_2 in states $k \in \mathcal{G}_i$. Let S_r denote the "ready" arms in states $k \in \mathcal{I}_i$, and let S_n denote the set of arms that are not "ready". There are two cases. If $|S_g \cup S_r| \ge M$, then some $S_p \subseteq S_g \cup S_r$ with $|S_p| = M$ is being played.

$$\Delta R_T + \mathbf{E}[\Delta \Phi_T] \ge \sum_{i \in S_p} (\lambda + h_i) \ge M\lambda \ge OPT/2$$

Next, if $|S_q \cup S_r| < M$, then all these arms are being played.

$$\Delta R_T + \mathbf{E}[\Delta \Phi_T] = \sum_{i \in S_q \cup S_r} (\lambda + h_i) + \sum_{i \in S_n} h_i \ge \sum_{i \in S_q \cup S_r \cup S_n} h_i = \sum_i h_i \ge OPT/2$$

Since the potentials of the arms not being played do not decrease (since all $h_l > 0$), the total change in reward plus potential is at least OPT/2.

Theorem 5.4. The BALANCEDINDEX policy in Figure 13 is a 2 approximation for MONOTONE bandits with multiple simultaneous plays of variable duration.

6 MONOTONE **Bandits: Switching Costs**

In several scenarios, playing an arm continuously incurs no extra cost, but switching to a different arm incurs a closing cost for the old arm and a setup cost for the new arm. For the applications mentioned in Section 1, in the context of UAV navigation [40], this is the cost of moving the UAV to the new location; or in the case of wireless channel selection, this is the setup cost of transmitting on the new channel.

We now show a 2-approximation for MONOTONE Bandits when the cost of switching out of arm i is c_i and the cost of switching into arm i is s_i . This cost is *subtracted* from the reward. Note that the switching cost depends additively on the closing and setup costs of the old and new arms. The remaining formulation is the same as Section 4.

Since the overall policy and proof are very similar to the version without these costs, we only outline the differences. First, we define the following variables: Let x_{kt}^i denote the probability of the event that arm i in state k is played after t steps and this arm was switched into from a different arm. Let y_{kt}^i denote the equivalent probability when the previous play was for the same arm. The LP relaxation is as follows:

Maximize
$$\sum_{i=1}^{n} \sum_{k \in \mathcal{S}_{i}} \sum_{t \in \mathcal{W}_{k}^{i}} \left(r_{k}^{i} \left(x_{kt}^{i} + y_{kt}^{i} \right) - \left(c_{i} + s_{i} \right) x_{kt}^{i} \right)$$

$$\sum_{i=1}^{n} \sum_{k \in \mathcal{S}_{i}} \sum_{t \in \mathcal{W}_{k}^{i}} x_{kt}^{i} + t y_{kt}^{i} \leq 1$$
(LPSWITCH)

The balanced dual is the following. (Recall the definition of $\Delta P_k^i(t)$ from Equation 2.)

$$\begin{array}{lll} \text{Minimize} & \lambda + \sum_{i=1}^n h_i & \text{(DUALSWITCH)} \\ \\ \lambda + th_i & \geq & r_k^i - c_i - s_i + f_k^i(t) \Delta P_k^i(t) & \forall i, k, t \\ t(\lambda + h_i) & \geq & r_k^i + f_k^i(t) \Delta P_k^i(t) & \forall i, k, t \\ \lambda & = & \sum_{i=1}^n h_i \\ \lambda, h_i & \geq & 0 & \forall i \end{array}$$

The proof of the next claim follows from complementary slackness exactly as the proof of Lemma 4.2.

Lemma 6.1. In the optimal solution to (DUALSWITCH), one of the following is true for every arm i with $h_i > 0$: Either repeatedly playing the arm yields per-step reward at least $\lambda + h_i$; or for every state $k \in S_i$, there exists $t \in W_k^i$ such that one of the following two LP constraints is tight with equality:

1.
$$\lambda + th_i \ge r_k^i - c_i - s_i + f_k^i(t)\Delta P_k^i(t)$$
.

2.
$$t(\lambda + h_i) \ge r_k^i + f_k^i(t)\Delta P_k^i(t)$$
.

Only consider arms with $h_i > 0$. The next lemma is similar to Lemma 4.3:

Lemma 6.2. For any arm i and state $k \in S_i$, if $\Delta P_k^i(t) < 0$, then: $\lambda + h_i = r_k^i + f_k^i(1)\Delta P_k^i(t)$.

For arm i, let t_k^i denote the smallest t for which some dual constraint for state k (refer Lemma 6.1) is tight. The state $k \in S_i$ belongs to G_i if the second constraint in Lemma 6.1 is tight at $t = t_k^i$, i.e.:

$$t_k^i(\lambda + h_i) = r_k^i + f_k^i(t_k^i)\Delta P_k^i(t) \tag{7}$$

By Lemma 6.2, this includes the case where $\Delta P_k^i(t) < 0$, so that $t_k^i = 1$.

Otherwise, the first constraint in Lemma 6.1 is tight at $t = t_k^i$. This state k belongs to \mathcal{I}_i , and becomes "ready" after t_k^i steps.

With these definitions, the BALANCEDINDEX policy is as follows: Stick with an arm i as long as its state is some $k \in \mathcal{G}_i$, and play it after waiting $t_k^i - 1$ steps. Otherwise, play any "ready" arm. If no "good" or "ready" arm is available, then idle.

Theorem 6.3. The BALANCEDINDEX policy is a 2-approximation for MONOTONE bandits with switching costs.

Proof. The definitions of the potentials and proof are the same as Lemma 4.4. The only difference is that the potential of state $k \in \mathcal{G}_i$ is defined to be fixed at p_k^i . Whenever the player sticks to arm i in state $k \in \mathcal{G}_i$ and plays it after waiting $t_k^i - 1$ steps, the reward plus change in potential amortized over the t_k^i steps (waiting plus playing) is exactly $\lambda + h_i$ by Eq. (7). The rest of the proof is the same as before.

7 FEEDBACK MAB with Observation Costs

In wireless channel scheduling, the state of a channel can be accurately determined by sending probe packets that consume energy. However, data transmission at high bit-rate yields only delayed feedback about channel quality. This aspect can be modeled by decoupling observation about the state of the arm via probing, from the process of utilizing or playing the arm to gather reward (data transmission). We model this as a variant of the FEEDBACK MAB problem, where at any step, M arms can be played without observing its state, and the reward of the underlying state is deposited in a bank. Further, any arm can be probed by paying a cost to determine its underlying state, and multiple such probes are allowed per step. The goal is to maximize the difference between the time average reward and probing cost. A version of the probe problem was first proposed in a preliminary draft of [27].

Formally, we consider the following variant of the FEEDBACK MAB problem. As before, the underlying 2-state Markov chain (on states $\{g,b\}$)) corresponding to an arm evolves irrespective of whether the arm is played or not. When arm i is played, a reward of r_i or 0 (depending on whether the underlying state is g or b respectively) is deposited into a bank. Unlike the FEEDBACK MAB problem, the player does not get to know the reward value or the state of the arm. However, during the end of any time step, the player can probe any arm i by paying cost c_i to observe its underlying state. We assume that the probes are at the end of a time step, and the state evolves between the probe and the start of the next time step. More than one arm can be probed and observed any time step, but at most M arms can be played, and the plays are of unit duration. The goal as before is to maximize the infinite horizon time-average difference between the reward obtained from playing the arms and the probing cost spent. Denote the difference between the reward and the probing cost as the "value" of the policy.

Though the probe version is not a MONOTONE bandit problem, we show that the above techniques can indeed be used to construct a policy which yields a $2 + \epsilon$ -approximation for any fixed $\epsilon > 0$.

7.1 LP Formulation

Let OPT denote the value of the optimal policy. The following is now an LP relaxation for the optimal policy. Let x_{gt}^i (resp. x_{bt}^i) denote the probability that arm i was last observed to be in state g (resp. b) t time steps ago and played at the current time step. Let z_{gt}^i (resp. z_{bt}^i) denote the probability that arm i was last observed to be in state g (resp. b) t steps ago and is probed at the current time step. The probes are at the end of a time step, and the state evolves between the probe and the start of the next time step. The LP formulation is as follows, as before the LP can be solved upto a $1 + \epsilon$ factor.

$$\begin{split} & \text{Maximize } \sum_{i=1}^{n} \sum_{t \geq 1} \left(r_i (u_{it} x_{gt}^i + v_{it} x_{bt}^i) - c_i (z_{gt}^i + z_{bt}^i) \right) \\ & \sum_{i=1}^{n} \sum_{t \geq 1} x_{gt}^i + x_{bt}^i & \leq & M \\ & \sum_{t \geq 1} t (z_{gt}^i + z_{bt}^i) & \leq & 1 & \forall i \\ & x_{st}^i & \leq & \sum_{l \geq t} z_{sl}^i & \forall i, t \geq 1, s \in \{g, b\} \\ & \sum_{t \geq 1} (1 - u_{it}) z_{gt}^i & = & \sum_{t \geq 1} v_{it} z_{bt}^i & \forall i \\ & x_{st}^i, z_{st}^i & \geq & 0 & \forall i, t \geq 1, s \in \{g, b\} \end{split}$$

The dual assigns a variable $\phi^i_{st} \geq 0$ for each arm i, state $s \in \{g,b\}$, and last observed time $t \geq 1$. It further assigns variables $h_i, p_i \geq 0$ per arm i, and $\lambda \geq 0$ globally. Let R^i_{st} be the expected reward of playing arm i in state s when last observed time is t. ($R^i_{gt} = r_i u_{it}$, $R^i_{gt} = r_i v_{it}$.) The balanced dual is as follows:

$$\begin{aligned} & \text{Minimize } M\lambda + \sum_i h_i \\ \lambda + \phi^i_{st} & \geq & R^i_{st} & \forall i, t \geq 1, s \in \{g, b\} \\ th_i & \geq & -c_i - (1 - u_{it})p_i + \sum_{l \leq t} \phi^i_{gl} & \forall i, t \geq 1 \\ th_i & \geq & -c_i + v_{it}p_i + \sum_{l \leq t} \phi^i_{bl} & \forall i, t \geq 1 \\ M\lambda & = & \sum_i h_i \\ \lambda, h_i, p_i, \phi^i_{st} & \geq & 0 & \forall i. s \in \{g, b\} \end{aligned}$$

We omit explicitly writing the corresponding primal. Note now that in the dual optimal solution, $\phi^i_{st} = \max(0, R^i_{st} - \lambda)$, $s \in \{g, b\}$. (This is the smallest value of ϕ^i_{st} satisfying the first constraint, and whenever we reduce ϕ^i_{st} , we preserve the latter constraints while possibly reducing h_i .) Moreover, we have the following complementary slackness conditions:

1.
$$h_i > 0 \Rightarrow \sum_{t > 1} t(z_{qt}^i + z_{bt}^i) = 1 + \omega > 0.$$

2.
$$z_{qt}^i > 0 \Rightarrow th_i = -c_i - (1 - u_{it})p_i + \sum_{l < t} \phi_{ql}^i$$
.

3.
$$z_{bt}^i > 0 \Rightarrow th_i = -c_i + v_{it}p_i + \sum_{l < t} \phi_{bl}^i$$

Lemma 7.1. Focus only on arms for which $h_i > 0$. For these arms, we have the following.

- 1. For at least one $t \ge 1$, $z_{qt}^i > 0$, and similarly, for some (possibly different) t, $z_{bt}^i > 0$.
- 2. Let $d_i = \min\{t \geq 1, z_{bt}^i > 0\}$, then $d_i h_i = -c_i + v_{id_i} p_i + \sum_{l \leq d_i} \phi_{bt}^i$. Further, define $m_i = |\{\phi_{bl}^i > 0 : l \leq d_i\}|$, then $\phi_{bl}^i > 0$ for $d_i m_i + 1 \leq l \leq d_i$ and $\phi_{bl}^i = 0$ for $l \leq d_i m_i$.
- 3. Let $e_i = \min\{t \geq 1, z_{gt}^i > 0\}$. Then, for all $t \leq e_i$, $\lambda + \phi_{gt}^i = R_{gt}^i$. Moreover, $e_i(\lambda + h_i) = \sum_{t \leq e_i} R_{gt}^i c_i (1 u_{ie_i})p_i$.

Proof. For part (1), by complementary slackness and using $h_i > 0$, we have $\sum_{t \geq 1} t(z^i_{gt} + z^i_{bt}) > 0$. But if $z^i_{gt} > 0$ for some t, then by $\sum_{t \geq 1} (1 - u_{it}) z^i_{gt} = \sum_{t \geq 1} v_{it} z^i_{bt}$, we have $z^i_{bt} > 0$ for some (possibly different) t. The reverse holds as well.

Part (2) follows by complementary slackness on $z^i_{bd_i} > 0$. The second part follows from the fact that ϕ^i_{bl} is non-decreasing since R^i_{bl} is non-decreasing.

For part (3), since $z^i_{be_i}>0$, by complementary slackness, $e_ih_i=-c_i-(1-u_{ie_i})p_i+\sum_{t\leq e_i}\phi^i_{gt}$. Note that $\phi^i_{gt}=\max(0,R^i_{gt}-\lambda)$. If $e_i=1$, then since the LHS is positive, it must be that $\phi^i_{gt}>0$, which implies that $\phi^i_{gt}=R^i_{gt}-\lambda$. If $e_i>1$, then we subtract $(e_i-1)h_i\geq -c_i-(1-u_{i(e_i-1)})p_i+\sum_{t\leq e_i-1}\phi^i_{gt}$ from the equality and get $h_i\leq (u_{ie_i}-u_{ie_i-1})p_i+\phi^i_{ge_i}$. The LHS is positive and the first term of the RHS is negative, so $\phi^i_{ge_i}>0$. Since ϕ^i_{gt} by the above formula is non-increasing, $\phi^i_{gt}>0$ $\forall t\leq e_i$. This in turn implies that $\phi^i_{gt}=R^i_{gt}-\lambda$ for all $t\leq e_i$. Substituting this back into the equality yields the second result.

7.2 Index Policy

Let the set of arms with $h_i > 0$ be S, we ignore all arms except those in S. The policy uses the parameters e_i , d_i and m_i defined in Lemma 7.1, If arm i was observed to be in state b, we denote it "not ready" for the next $d_i - m_i$ steps, and denote it to be "ready" at the end of the $(d_i - m_i)^{th}$ step.

```
Constraint: The policy keeps initiating "Stage 1" with ready arms if less than M arms are in either stages.

Stage 1: /* Arm i is last observed to be b but has turned "ready" */

1. Try. Play the arm for the next m_i steps.

2. Probe the arm at the end of m_i^{th} step.

If state is g, then go to Stage 2 for that arm.

Stage 2: /* Arm i just observed to be in state g */

1. Exploit. Play the arm for the next e_i steps.

2. Probe the arm at the end of e_i^{th} step.

If state is g then go to Step (1) of Stage (2).
```

Figure 14: The policy for FEEDBACK MAB with observations.

Theorem 7.2. The policy in Fig. 14 is a $2 + \epsilon$ approximation to FEEDBACK MAB with observation costs.

Proof. Let OPT denote the $1+\epsilon$ approximate LP solution. Recall that $\phi^i_{st} = \max(0, R^i_{st} - \lambda)$, $s \in \{g, b\}$. Define the following potentials for each arm i. If it was last observed to be in state b some t steps ago, define its potential to be $(\min(t, d_i - m_i))h_i$; if it was last observed in state g, define its potential to be p_i . We show that the time-average expected value (reward minus cost) plus change in potential per step is at least $\min(M\lambda, \sum_{i \in S} h_i) \geq OPT/2$. Since the potentials are bounded, this proves a 2-approximation.

Each ready arm in Stage 1 is played for m_i steps and probed at the end of the m_i^{th} step. Suppose that the arm was last observed to be in state b some t steps ago. The total expected value is $-c_i + \sum_{l=t}^{t+m_i-1} R_{bl}^i$, which is at least $-c_i + \sum_{l=d_i-m_i+1}^{d_i} R_{bl}^i$ since $R_{bl}^i = r_i v_{il}$ is non-decreasing in l. The expected change in potential is $v_{i(t+m_i-1)}p_i - (d_i-m_i)h_i$, since the arm loses the potential build up of $(d_i-m_i)h_i$ while it was not ready, and has a probability of $v_{i(t+m_i-1)}$ of becoming good. This is at least $v_{id_i}p_i - (d_i-m_i)h_i$ since by definition, $t \geq d_i - m_i + 1$. After m_i steps, the total expected value plus change in potential is at least $-c_i + \sum_{l=d_i-m_i+1}^{d_i} R_{bl}^i + v_{id_i}p_i - (d_i-m_i)h_i \geq m_ih_i + \sum_{l=d_i-m_i+1}^{d_i} (R_{bl}^i - \phi_{bl}^i)$. The inequality follows by Lemma 7.1 Part (2). Since $R_{bl}^i - \phi_{bl}^i = \lambda$ for $d_i - m_i + 1 \leq l \leq d_i$, the total expected change in value plus potential is $m_i(\lambda + h_i)$. Thus, the average per step for the duration of the plays is at least $\lambda + h_i$. (This proof also shows that if $m_i = 0$, then the probing on the previous step does not decrease the potential.)

Similarly, each arm i in Stage 2 was probed and found to be good, so that it is exploited for e_i steps and probed at the end of the e_i^{th} step. During these e_i steps, the total expected value is $\sum_{t \leq e_i} R_{gt}^i - c_i$, and expected change in potential is $-(1-u_{ie_i})p_i$, since the arm has probability $(1-u_{ie_i})$ of being in a bad state at the end. By Lemma 7.1, Part (3), the total expected value plus change in potential is $\sum_{t \leq e_i} R_{gt}^i - c_i - (1-u_{ie_i})p_i = e_i(\lambda + h_i)$, so the average change per step is $\lambda + h_i$.

Now, if M arms are currently in Stage 1 or 2, then the total value plus change in potential for these arms is at least $M\lambda \geq OPT/2$. If fewer than M arms are in those stages, then every arm i that is not in Stage 1 or Stage 2 is in state b and not "ready". Thus, its change in potential is h_i . Moreover, for every arm j that is in Stage 1 or Stage 2, we also get a contribution of at least $\lambda + h_j \geq h_j$. Summing, we get a expected value plus change in potential of at least $\sum_{i \in S} h_i \geq OPT/2$, which completes the proof.

8 Non-Preemptive Machine Replenishment

Finally, we show our technique of balancing provides a 2-approximation for an unrelated, yet classic, restless bandit problem [10, 23, 39]: modeling breakdown and repair of machines. Interestingly, we also show that the Whittle index policy is an arbitrarily poor approximation to non-preemptive machine replenishment, and thus the technique we suggest can be significantly stronger than the Whittle index policies.

There are n independent machines whose performance degrades with time in a Markovian fashion. This is modeled by transitions between states yielding decreasing rewards. At any step, any machine can be moved to a repair queue by paying a cost. The repair process is non-preemptive, Markovian, and can work on at most M machines per time step. A scheduling policy decides when to move a machine to a repair queue and which machine to repair at any time slot. The goal is to find a scheduling policy to maximize the time-average difference between rewards and repair cost. Note that if an arm is viewed as a machine, playing it corresponds to repairing it, and does not yield reward. In that sense, this problem is like an inverse of the Monotone bandits problem. We emphasize that the repairs are *non-preemptive*, which means that once a repair is started, it cannot be stopped.

Formally, there are n machines. Let S_i denote the set of active states for machine i. If the state of machine i is $u \in S_i$ at time the beginning of time t, the state evolves into $v \in S_i$ at time t+1 w.p. \mathbf{p}_{uv} . The state transitions for different machines when they are active are independent. If the state of machine i is $u \in S_i$ during a time step, it accrues reward $r_u \geq 0$. We assume each S_i is poly-size.

During any time instant, machine i in state $u \in \mathcal{S}_i$ can be scheduled for maintenance by moving it to the repair queue starting with the next time slot by paying cost c_u . The maintenance process for machine i takes time which is distributed as $\texttt{Geometric}(s_i)$, independent of the other machines. Therefore, if the repair process works on machine i at any time step, this repair completes after that time step with probability s_i . During the time when the machine is in the repair queue, it yields no reward. When the machine is in the repair queue, we denote its state by κ_i . The maintenance process is non-preemptive, and the server can maintain at most M machines at any time. When a repair completes, the machine i returns to its "initial active state" $\rho_i \in \mathcal{S}_i$ at the beginning of the next time slot. The goal is to design a scheduling policy so that the time- average reward minus maintenance cost is maximized.

In related work, Munagala and Shi [39] showed using a novel queuing analysis that when the repair process is *preemptive*, M=1, and when $\mathcal{S}_i=\{\rho_i,b_i\}$ for all machines i, and $r_{b_i}=0$, so that the machine is either "active" (state ρ_i) or "broken" (state b_i), then a simple greedy policy that is equivalent to the Whittle index policy is a 1.51 approximation. However, as we show later, the Whittle index policy can be arbitrarily bad for non-preemptive repairs since it computes indices for each machine separately. We now show that our duality-based technique yields a 2-approximation policy with general \mathcal{S}_i , M, and non-preemptive repairs.

8.1 LP Formulation and Dual

We now present an LP bound on the optimal policy. For any policy, let x_u denote the steady state probability that machine i is in state u during a time step, and z_u denote the steady state probability that the machine i transitions from state $u \in \mathcal{S}_i$ to state κ_i . We assume the policy moves a machine to the repair queue at the beginning of a time slot, and that repairs finish at end of a time slot. Note that it does not make sense to repair a machine in its initial state x_{ρ_i} so $z_{\rho_i} = 0$.

Maximize
$$\sum_{i} \sum_{u \in \mathcal{S}_i} (r_u x_u - c_u z_u)$$

$$\sum_{i} x_{\kappa_{i}} \leq M$$

$$x_{\kappa_{i}} + \sum_{u \in \mathcal{S}_{i}} x_{u} \leq 1 \qquad \forall i$$

$$\sum_{v \in \mathcal{S}_{i}, v \neq u} x_{v} \mathbf{p}_{vu} = z_{u} + \sum_{v \in \mathcal{S}_{i}, v \neq u} x_{u} \mathbf{p}_{uv} \quad \forall i, u \in \mathcal{S}_{i} \setminus \{\rho_{i}\}$$

$$s_{i} x_{\kappa_{i}} + \sum_{v \in \mathcal{S}_{i}, v \neq u} x_{v} \mathbf{p}_{v\rho_{i}} = \sum_{v \in \mathcal{S}_{i}, v \neq \rho_{i}} x_{\rho_{i}} \mathbf{p}_{\rho_{i}v} \quad \forall i$$

$$z_{u}, x_{u} \geq 0 \quad \forall i, u \in \mathcal{S}_{i} \cup \{\kappa_{i}\}$$

The dual of the above LP assigns potentials ϕ_u for each state $u \in S_i$. Further, it assigns a value $h_i \ge 0$ for each machine i, and a global variable $\lambda \ge 0$. We directly write the balanced dual:

$$\begin{aligned} & \text{Minimize } M\lambda + \sum_{i} h_{i} \\ \lambda + h_{i} & \geq & s_{i}\phi_{\rho_{i}} & \forall i \\ h_{i} & \geq & r_{u} + \sum_{v \in \mathcal{S}_{i}} \mathbf{p}_{uv}(\phi_{v} - \phi_{u}) & \forall i, u \in \mathcal{S}_{i} \\ \phi_{u} + c_{u} & \geq & 0 & \forall i, u \in \mathcal{S}_{i} \\ M\lambda & = & \sum_{i} h_{i} \\ \lambda, h_{i} & > & 0 & \forall i \end{aligned}$$

Note that $M\lambda = \sum_i h_i \ge OPT/2$. We omit explicitly writing the corresponding primal formulation. Now, Focus only on machines for which $h_i > 0$. We have the following complementary slackness conditions:

1.
$$h_i > 0 \Rightarrow x_{\kappa_i} + \sum_{u \in \mathcal{S}_i} x_u = 1 - \omega > 0$$

2.
$$x_u > 0 \Rightarrow h_i = r_u + \sum_{v \in S_i} \mathbf{p}_{uv} (\phi_v - \phi_u)$$
.

3.
$$z_u > 0 \Rightarrow \phi_u + c_u = 0$$
.

4.
$$x_{\kappa_i} > 0 \Rightarrow \lambda + h_i = s_i \phi_{\rho_i}$$
.

8.2 Index Policy and Analysis

Scheduling: Consider only machines i with $h_i > 0$

If the machine in in state u where $z_u > 0$, then move the machine to repair queue.

Repair: Service any subset S_w of at most M machines in the repair queue non-preemptively. w.p. s_i , the service for $i \in S_w$ completes and it moves to state ρ_i .

Figure 15: The repair policy from our LP-duality approach

Claim 8.1. Consider only machines with $h_i > 0$. There are two cases:

- 1. For machines in which $z_v > 0$ for some v, we have $x_{\kappa_i} > 0$ so that the policy can only reach states $u \in S_i$ in which $x_u + z_u > 0$.
- 2. For machines in which $z_v = 0$ for all v, we have $x_{\kappa_i} = 0$. The policy will never repair the machine, and after a finite number of steps,, the machine will only visit states $u \in \mathcal{S}_i$ for which $x_u > 0$.

Proof. Adding the third and fourth constraints of the primal yields $s_i x_{\kappa_i} = \sum_{u \in \mathcal{S}_i} z_u$. If for some $v, z_v > 0$, then $x_{\kappa_i} > 0$, which by the fourth constraint in the primal implies that $x_{\rho_i} > 0$. Now, suppose that $x_v > 0$, then for every state u such that $\mathbf{p}_{vu} > 0$, the third constraint in the primal implies that $z_u + x_u > 0$. If $z_u > 0$, then the policy will stop at state u and enter machine i into the repair queue. If $z_u = 0$, then it must

be that $x_u > 0$. Repeatedly using the above argument starting at $v = \rho_i$, we see that the policy will only visit states with $x_u + z_u > 0$, not going beyond the first state where $z_u > 0$.

For machines in which $z_v = 0$ for all v, conditions (3) and (4) in the primal imply that $\{x_v\}$ are the steady state probabilities of a Markov chain with transition matrix $[\mathbf{p}_{uv}]$. Therefore, after a finite number of steps, the machine will only go to states $u \in \mathcal{S}_i$ for which $x_u > 0$.

Theorem 8.2. The policy in Fig. 15 is a 2-approximation for non-preemptive machine replenishment.

Proof. We next interpret ϕ_u as the potential for state $u \in \mathcal{S}_i$. Let the potential for state κ_i be 0. We show that in each step, the expected reward plus change in potential is at least OPT/2.

First, when any active machine i enters a state u with $z_u > 0$, then the machine is moved to the repair queue by paying cost c_u . The potential change is $-\phi_u$, and the sum of the cost and potential change is $-c_u - \phi_u$. The last term is 0 by complementary slackness. Therefore, moving a machine to the repair queue does not alter the potential.

Next, let S_r denote the set of machines in the repair queue, and let $S_w \subseteq S_r$ denote the subset of these machines being repaired at the current time. Note that if $|S_r| < M$, then $S_w = S_r$, otherwise $|S_w| = M$. For each machine $i \in S_w$, the repair finishes w.p. s_i , and the machine's potential changes by ϕ_{ρ_i} . Therefore, the expected change in potential per step is $s_i \phi_{\rho_i} = \lambda + h_i$ by complementary slackness.

Suppose first that $|S_w|=M$, then the net reward plus change in potential is at least $M(\lambda+h_i)>M\lambda\geq OPT/2$. Suppose that $|S_w|< M$, then must have $S_w=S_r$. Note that any machine that enters a state u with $z_u>0$ will be automatically moved to S_r at the beginning of the time step. Using this along with Claim 8.1, we have that after a finite number of steps, any machine $i\notin S_r$ enters a state u with $x_u>0$. (Since we care about infinite horizon average reward, the finite number of steps don't matter.) The reward plus change in potential for machine $i\notin S_r$ is $r_u+\sum_{v\in S_i}\mathbf{p}_{uv}(\phi_v-\phi_u)=h_i$ by complementary slackness. Therefore, the total reward plus change in potential is $\sum_{i\in S_r}(\lambda+h_i)+\sum_{i\notin S_r}h_i\geq \sum_i h_i\geq OPT/2$. Since the potentials are bounded, the policy is a 2- approximation.

8.3 Gap of the Whittle Index

We now show that the Whittle index policy is an arbitrarily poor approximation to non-preemptive machine replenishment. Note that in the situation shown below, Whittles index is a 1.51 approximation when repairs can be preempted [39]. However, when no preemption is allowed, the policy can perform arbitrarily poorly.

Theorem 8.3. The Whittle index policy is an arbitrarily poor approximation for non-preemptive machine replenishment even with 2 machines and M = 1 repairs per step.

Proof. Suppose M=1, there are two machines $\{1,2\}$, and $\mathcal{S}_i=\{\rho_i,b_i\}$ for machines $i\in\{1,2\}$. Let $r_{\rho_i}=r_i$ and let $r_{b_i}=0$, so that the machine is either "active" (state ρ_i) or "broken" (state b_i). Let p_i denote the probability of transitioning from state ρ_i to b_i . Assume $c_i=0$. Note that playing a machine corresponds to moving it to the repair queue.

The Whittle index of a state is the largest penalty that can be charged per maintenance step so that the optimal single machine policy will still schedule the machine for maintenance on entering the current state. In 2-state machines mentioned above, the Whittle index in state ρ_i is negative, since even with penalty zero per repair step, the policy will not schedule the machine for maintenance in the good state. The Whittle index for state b_i is $\eta_i = s_i r_i/p_i$, since for this value of penalty, the expected reward of r_i/p_i per renewal is the same as the expected penalty of η_i/s_i paid for maintenance in the renewal period.

Suppose $s_1=1/n^4$, $s_2=1$, $p_1=1/n$, $p_2=1$, $r_1=r_2=1$. If used by itself, machine 1 yields reward $r_1\frac{s_1}{s_1+p_1}\approx 1/n^3$ and machine 2 yields reward $r_2\frac{s_2}{s_2+p_2}=1$. Any reasonable policy will therefore only maintain machine 2 and ignore machine 1. However, in the Whittle index policy, when machine 1 is

broken and machine 2 is active, the policy decides to maintain machine 1 (since the Whittle index, η_1 , of b_1 is positive and that of ρ_2 is negative). In this case, machine 1 is scheduled for repair. This repair takes $O(n^4)$ time steps and cannot be interrupted. Moreover, since machine 2 is bad at least half the time, this "blocking" by machine 1 will happen with rate O(1/2), so in the long run, machine 2 is almost always broken and the Whittle index policy obtains reward $O(1/n^3)$, while the optimal policy obtains reward $O(1/n^3)$ is only maintaining machine 2.

9 Open Questions

Our work throws open interesting research avenues. First, can our algorithmic techniques be extended to other subclasses of restless bandits, for instance, the POMDP problem obtained by generalizing FEEDBACK MAB to K>2 states per arm? Note that unlike the K=2 case considered here, the transition probability values are no longer monotone as they are based on an underlying Markov chain. Next, can matching hardness results be shown for these problems, particularly FEEDBACK MAB. Finally, our analysis effectively uses piece-wise linear Lyapunov functions. Such functions derived from LP relaxations have also been used by Bertsimas, Gamarnik, and Tsitsiklis [11] to show stability in multi-class queuing systems. Though the techniques and results in that work are very different from ours, it would be interesting to explore whether our techniques extend to multi-class queuing problems.

Acknowledgment. We thank Shivnath Babu, Jerome Le Ny, Ashish Goel, and Alex Slivkins for discussions concerning parts of this work. We also thank the anonymous FOCS 2007 and SODA 2009 reviewers for several helpful comments on earlier drafts of this work.

References

- [1] D. Adelman and A. J. Mersereau. Relaxations of weakly coupled stochastic dynamic programs. *Operations Research*, 2008.
- [2] K. J. Arrow, D. Blackwell, and M. A. Girshick. Bayes and minmax solutions of sequential decision problems. *Econometrica*, 17:213–244, 1949.
- [3] P. S. Ansell, K. D. Glazebrook, J. E. Niño-Mora, and M. OKeeffe. Whittle's index policy for a multi-class queueing system with convex holding costs. *Math. Meth. Oper. Res.*, 57:21–39, 2003.
- [4] M. Asawa and D. Teneketzis. Multi-armed bandits with switching penalties. *IEEE Transactions on Automatic Control*, 41(3):328–348, 1996.
- [5] P. Auer. Using confidence bounds for exploitation-exploration trade-offs. *Journal of Machine Learning Research*, 3:397–422, 2002.
- [6] P. Auer, N. Cesa-Bianchi, and P. Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine Learning*, 47(2-3):235–256, 2002.
- [7] P. Auer, N. Cesa-Bianchi, Y. Freund, and R. E. Schapire. Gambling in a rigged casino: The adversarial multi-armed bandit problem. In *Proc. IEEE Symp. on Foundations of Computer Science*, 1995.
- [8] J. S. Banks and R. K. Sundaram. Switching costs and the gittins index. *Econometrica*, 62(3):687–694, 1994.
- [9] A. Bar-Noy, R. Bhatia, J. Naor, and B. Schieber. Minimizing service and operation costs of periodic scheduling. In *Proc. ACM-SIAM Symp. on Discrete Algorithms*, pages 11–20, 1998.
- [10] D. Bertsekas. Dynamic Programming and Optimal Control. Athena Scientific, second edition, 2001.
- [11] D. Bertsimas, D. Gamarnik, and J. N. Tsitsiklis. Performance of multiclass markovian queueing networks via piecewise linear lyapunov functions. *Annals of Applied Probability*, 11(4):1384–1428, 2001.

- [12] D. Bertsimas and J. Nino-Mora. Conservation laws, extended polymatroids and multi-armed bandit problems: A unified polyhedral approach. *Math. of Oper. Res.*, 21(2):257–306, 1996.
- [13] D. Bertsimas and J. Niño-Mora. Restless bandits, linear programming relaxations, and a primal-dual index heuristic. *Oper. Res.*, 48(1):80–90, 2000.
- [14] M. Brezzi and T-L. Lai. Optimal learning and experimentation in bandit problems. *J. Econ. Dyn. Control*, 27(1):87–108, 2002.
- [15] N. Cesa-Bianchi, Y. Freund, D. Haussler, D. P. Helmbold, R. E. Schapire, and M. K. Warmuth. How to use expert advice. *J. ACM*, 44(3):427–485, 1997.
- [16] D. P. de Farias and N. Megiddo. Combining expert advice in reactive environments. J. ACM, 53(5):762–799, 2006.
- [17] A. Flaxman, A. Kalai, and H. B. McMahan. Online convex optimization in the bandit setting: Gradient descent without a gradient. In *Annual ACM-SIAM Symp. on Discrete Algorithms*, 2005.
- [18] J. C. Gittins and D. M. Jones. A dynamic allocation index for the sequential design of experiments. *Progress in statistics (European Meeting of Statisticians)*, 1972.
- [19] K. D. Glazebrook and H. M. Mitchell An index policy for a stochastic scheduling model with improving/deteriorating jobs. *Naval Research Logistics*, 49:706–721, 2002.
- [20] K. D. Glazebrook, H. M. Mitchell, and P. S. Ansell. Index policies for the maintenance of a collection of machines by a set of repairmen. *European J. Oper. Res.*, 165(1):267–284, 2005.
- [21] K. D. Glazebrook, D. Ruiz-Hernandez, and C. Kirkbride. Some indexable families of restless bandit problems. *Advances in Applied Probability*, 38:643–672, 2006.
- [22] A. Goel, S. Guha, and K. Munagala. Asking the right questions: Model-driven optimization using probes. In *Proc. of the 2006 ACM Symp. on Principles of Database Systems*, 2006.
- [23] K. Goseva-Popstojanova and K. Trivedi. Stochastic modeling formalisms for dependability, performance and performability. In *Performance Evaluation: Origins and Directions*, pages 403–422, 2000.
- [24] S. Guha and K. Munagala. Approximation algorithms for budgeted learning problems. In *Proc. ACM Symp. on Theory of Computing (STOC)*, 2007.
- [25] S. Guha and K. Munagala. Approximation algorithms for partial-information based stochastic control with Markovian rewards. In *Proc.* 48th IEEE Symp. on Foundations of Comp. Sci. (FOCS), 2007.
- [26] S. Guha and K. Munagala. Model-driven optimization using adaptive probes. In *Proc. ACM-SIAM Symp. on Discrete Algorithms*, 2007.
- [27] S. Guha, K. Munagala, and S. Sarkar. Optimizing transmission rate in wireless channels using adaptive probes. *SIGMETRICS/Performance*, 2006. See also *Computing Research Repository*, arXiv:0804.1724, 2008.
- [28] S. Guha, K. Munagala, and P. Shi. Approximation algorithms for restless bandit problems. In *Proc. ACM-SIAM Symp. on Discrete Algorithms*, 2009.
- [29] J. T. Hawkins. A Langrangian decomposition approach to weakly coupled dynamic optimization problems and its applications. PhD thesis, Operations Research Center, Massachusetts Institute of Technology, 2003.
- [30] T. Javidi, B. Krishnamachari, Q. Zhao, and M. Liu. Optimality of myopic sensing in multi-channel opportunistic access. In *Proc. IEEE International Conf. in Communications (ICC '08)*, 2008.
- [31] L. P. Kaelbling, M. L. Littman, and A. R. Cassandra. Planning and acting in partially observable stochastic domains. *Artificial Intelligence Journal*, 101:99–134, 1998.
- [32] S. M. Kakade and M. J. Kearns. Trading in markovian price models. In *COLT*, pages 606–620, 2005.
- [33] M. S. Kodialam and T. V. Lakshman. Achievable rate region for wireless systems with time varying channels. In *INFOCOM*, pages 53–61, 2007.

- [34] T. L. Lai and H. Robbins. Asymptotically efficient adaptive allocation rules. *Advances in Applied Mathematics*, 6:4–22, 1985.
- [35] R. Levi, M. Pál, R. Roundy, and D. B. Shmoys. Approximation algorithms for stochastic inventory control models. In *IPCO*, pages 306–320, 2005.
- [36] N. Littlestone and M. K. Warmuth. The weighted majority algorithm. Inf. Comput., 108(2):212–261, 1994.
- [37] K. Liu and Q. Zhao. A restless bandit formulation of multi-channel opportunistic access: Indexablity and index policy. *Computing Research Repository*, arXiv:0810.4658, 2008.
- [38] C. Lund and M. Yannakakis. The approximation of maximum subgraph problems. In *Proc.* 20th Intl. Colloq. on Automata, Languages and Programming (ICALP), pages 40–51, 1993.
- [39] K. Munagala and P. Shi. The stochastic machine replenishment problem. In *Proc.* 13th Conf. on Integer Programming and Combinatorial Optimization (IPCO), 2008.
- [40] J. Le Ny, M. Dahleh, and E. Feron. Multi-UAV dynamic routing with partial observations using restless bandits allocation indices. In *Proceedings of the 2008 American Control Conference*, 2008.
- [41] C. H. Papadimitriou and J. N. Tsitsiklis. The complexity of optimal queuing network control. *Math. Oper. Res.*, 24(2):293–305, 1999.
- [42] H. Robbins. Some aspects of the sequential design of experiments. *Bulletin American Mathematical Society*, 55:527–535, 1952.
- [43] A. Slivkins and E. Upfal. Adapting to a changing environment: The Brownian restless bandits. In COLT, 2008.
- [44] R. Smallwood and E. Sondik. The optimal control of partially ovservable Markov processes over a finite horizon. *Operations Research*, 21:1071–1088, 1971.
- [45] E. J. Sondik. The optimal control at partially observable Markov processes over the infinite horizon: Discounted costs. *Operations Research*, 26(2):282–304, 1978.
- [46] J. N. Tsitsiklis. A short proof of the Gittins index theorem. Annals of Appl. Prob., 4(1):194–199, 1994.
- [47] V. Vazirani. Approximation Algorithms. Springer, 2001.
- [48] A. Wald. Sequential Analysis. Wiley, New York, 1947.
- [49] R. R. Weber and G. Weiss. On an index policy for restless bandits. J. Appl. Prob., 27(3):637–648, 1990.
- [50] G. Weiss. Branching bandit processes. Probab. Engng. Inform. Sci., 2:269–278, 1988.
- [51] P. Whittle. Arm acquiring bandits. Ann. Probab., 9:284–292, 1981.
- [52] P. Whittle. Restless bandits: Activity allocation in a changing world. Appl. Prob., 25(A):287–298, 1988.
- [53] N. E. Young. The k-server dual and loose competitiveness for paging. Algorithmica, 11(6):525–541, 1994.
- [54] Q. Zhao, B. Krishnamachari, and K. Liu. On myopic sensing for multi-channel opportunistic access: Structure, optimality, and performance. *IEEE Trans. Wireless Communications*, 2008.

A Omitted Proofs

The following proofs are deferred here because they are independent of our duality based technique, and because of their lengths, we fear that they may detract from the paper's flow.

A.1 Proof of Theorem 2.1

We show examples in which the myopic policy and the optimal index policy exhibit the desired gaps against the optimum.

Gap of the Myopic Policy. We now show an instance where the myopic policy that plays the arm with the highest expected next-step reward has gap $\Omega(n)$ with respect to the reward of the optimal policy. The instance is an extension of the instance constructed above. There is one "type 1" deterministic arm with reward $r_1=1$. There are n i.i.d. "type 2" arms with $r_2=n$, $\beta=\frac{1}{2^n}$, and $\frac{\alpha}{\alpha+\beta}=\frac{1}{n}$.

First consider the myopic policy. Any policy encounters an instant where all the type 2 arms are in state b. In this case, the myopic next step reward of any of these arms is at most $r_2 \frac{\alpha}{\alpha + \beta} < 1$, so that the policy always plays the type 1 arm, yielding long-term reward of 1.

Next consider the myopic policy that ignores the type 1 arm. Such a policy performs round-robin on the arms when it observes all of them to be in state b. In this case, the probability that the arm it plays will be in state g is at least $v_n \geq \frac{1}{2^n}$. Therefore, the behavior of this policy is dominated by the following 2-state Markov chain: The two states are h and l; state h yields reward n, and state l, reward l. The transition probabilities from l to l and vice-versa are l l 1. The long-term reward is therefore at least l 2, which lower-bounds the reward of the optimal policy.

Non-optimality of Index policies. We will now show an instance where there is a constant factor gap between the optimal policy and the optimal index policy. The example has 3 arms. Arm 1 is deterministic with reward $r_1 = 1$. Arms 2 and 3 are *i.i.d.* with $\alpha = \beta = 0.1$ and reward in state g being $r_2 = 2$.

We compute the optimal policy by value iteration [10] using a discount factor of $\gamma=0.99$ (to ensure the dynamic program converges). The optimal policy always plays arms 2 or 3 if either was just observed in state g. The decisions are complicated only if both arms 2,3 were last observed in state b. In this case, we can compactly represent the current state by the pair $(k_1,k_2) \in \mathbf{Z}^+ \times \mathbf{Z}^+$, representing the time steps ago that arms 2,3 were observed in state b respectively. For such a state, the policy either plays arm 1; or plays arms 2 or 3 depending on whether $k_1 > k_2$ or not. Such a policy is therefore completely characterized by the region $\mathcal D$ on the (k_1,k_2) plane where the decision is to play arm 1; in the remaining region, it plays arm 2 or 3 depending on whether $k_1 > k_2$ or not. For the optimal policy, we have:

$$\mathcal{D}^* = \{(k_1, k_2) \in \mathbf{Z}^+ \times \mathbf{Z}^+ \mid k_1 \le 4, k_2 \le 4, k_1 + k_2 \le 6\}$$

In other words, the description of the optimal policy is as follows (note that it is symmetric w.r.t. arms 2, 3):

State (k_1, k_2)	Play Arm
$k_1 \le 3, k_2 < k_1$	1
$k_1 = 4, k_2 \le 2$	1
$k_1 = 4, k_2 = 3$	2
$k_1 = 4, k_2 > 4$	3
$k_1 \ge 5, k_2 < k_1$	2

Table 1: Description of the optimal policy when both arms 2, 3 were last observed to be b. Note that the policy is symmetric w.r.t. arms 2, 3, and furthermore, $k_1 \neq k_2$.

Note the following non-index behavior where given the state of arms 1 and 2, the decision to play switches between these arms depending on the state of arm 3. If arm 2 was observed to be b some 4 steps ago, then: (i) If arm 3 was b some 2 steps ago, the policy plays arm 1; (ii) If arm 3 was b some 3 steps ago, the policy plays arm 2. To compute the reward of this policy, we observe that it has an equivalent description as a Markov Chain over 6 states (these new states correspond to groups of states in the original process). A closed form evaluation of this chain shows that the reward of the optimal policy is 1.46218.

We next perform this evaluation for the nearby index policies. Note that for any index policy, the region \mathcal{D} has to be an axis-parallel square. The first is where the decision for $k_1=4, k_2\leq 2$ is to play arm 2, so that $\mathcal{D}=\{(k_1,k_2)|k_1,k_2\leq 3\}$. This policy evaluates to an average reward 1.46104. The next is where the decision for $k_1=4, k_2=3$ is to play arm 1, so that $\mathcal{D}=\{(k_1,k_2)|k_1,k_2\leq 4\}$. This policy has reward 1.46167. Other index policies have only worse reward. This implies that there is a constant factor gap between the optimal policy and the best index policy.

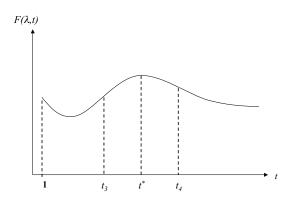


Figure 16: Behavior of the function $F(\lambda, t)$.

A.2 Proof of Theorem 2.6

Since the proof focuses on a single arm i, we omit the subscript for the arm. For notational convenience, denote $t^* = t(\lambda)$. The expression for $H(\lambda)$ follows easily from Lemma 2.5. Recall from Definition 3 that $F(\lambda,t) = R(t) - \lambda Q(t)$ as the value of policy $\mathcal{P}(t)$.

Case 1: $\lambda \geq r\left(\frac{\alpha}{\alpha+\beta(\alpha+\beta)}\right)$. Consider the subcase $\lambda \geq r$. The function $F(\lambda,t)$ is maximized by driving the expression (which is always non-positive) to zero. This happens when $t=\infty$. Otherwise, when $r>\lambda$ observe (using the upper bound of v_t) that

$$F(\lambda, t) = \frac{(r - \lambda)v_t - \lambda\beta}{v_t + t\beta} \le \frac{(r - \lambda)\frac{\alpha}{\alpha + \beta} - \lambda\beta}{v_t + t\beta}$$

The above is now non-positive, and it follows again that $t=\infty$ is the optimum solution.

Case 2: In this case let $\lambda = r\left(\frac{\alpha}{\alpha + \beta(\alpha + \beta)}\right) - \rho$ for some $\rho > 0$. Rewrite the above expression as

$$F(\lambda, t) = (r - \lambda) - \beta \frac{\lambda + t(r - \lambda)}{v_t + t\beta}$$

Define the following quantities (independent of t):

$$\nu = (1 - \alpha - \beta) \qquad \eta = \frac{\alpha}{\alpha + \beta} \log \frac{1}{\nu} \qquad \phi = \eta \lambda + \frac{\alpha}{\alpha + \beta} (r - \lambda)$$
$$\mu = \eta (r - \lambda) \qquad \omega = \lambda \beta - \alpha \frac{r - \lambda}{\alpha + \beta} = -\rho \frac{\alpha + \beta(\alpha + \beta)}{\alpha + \beta}$$

Observe $r - \lambda > \rho$. Note that $\phi, \mu \geq 0$. By assumption, the value $\nu \in (\delta, 1]$ has polynomial bit complexity. The same holds for η, ϕ, μ and ρ . Relaxing t to be a real, observe:

$$\frac{\partial F(\lambda, t)}{\partial t} = \frac{\beta}{(v_t + t\beta)^2} \left((\phi + \mu t) \nu^t + \omega \right)$$

Since the denominator of $\partial F/\partial t$ is always non-negative, the value of t^* is either $t^*=1$, or the point where the sign of the numerator $g(t)=(\phi+\mu t)\nu^t+\omega$ changes from + to -. We observe that g(t) has a unique maximum at $t_3=\frac{1}{\log(1/\nu)}-\frac{\phi}{\mu}$. If $g(t_3)$ is negative then the numerator of $\frac{\partial F(\lambda,t)}{\partial t}$ is always negative and the optimum solution is at $t^*=1$.

If $g(t_3)$ is positive, then it cannot change sign from + to - in the range $[1, t_3)$ since it has a unique maximum. Therefore in this range t = 1, $t = \lfloor t_3 \rfloor$, or $t = \lceil t_3 \rceil$ are the optimum solutions.

But for $t \ge t_3$ since g(t) is decreasing, $\partial F/\partial t$ changes sign once from + to - as t increases, and $\to 0$ as $t \to \infty$. This behavior is illustrated in Figure 16. Therefore, we find a $t_4 > t_3$ such that $g(t_4) < 0$, and perform binary search in the range $[t_3, t_4]$ to find the point where F is maximized. It is easy to compute t_4 with polynomial bit complexity in the complexities of ν, η, ϕ, μ and ρ . We finally compare this maximal value of F to the values of F at $1, |t_3|, [t_3]$. Thus we can solve $H(\lambda)$ and obtain t^* in polytime.

A.3 Proof of Theorem 2.12

We first show the structure of the optimal solution to (WHITTLE). Using the notation from Definition 3, we have: $H_i(\lambda) = R_i(\lambda) - \lambda Q_i(\lambda)$. Let $R(\lambda) = \sum_{i=1}^n R_i(\lambda)$ and $Q(\lambda) = \sum_{i=1}^n Q_i(\lambda)$. The following lemma shows that the optimal solution to (WHITTLE) is obtained by choosing λ such that $Q(\lambda) \approx 1$.

Lemma A.1. The optimal solution to Whittle's LP chooses a penalty λ^* and a fraction $a \in [0,1]$, so that $aQ(\lambda_-^*) + (1-a)Q(\lambda_+^*) = 1$. Here, $\lambda_-^* \le \lambda^* < \lambda_+^*$ with $|\lambda_+^* - \lambda_-^*| \to 0$. The solution corresponds to a convex combination of $\mathcal{P}_i(t_i(\lambda_-^*))$ with weight a and $\mathcal{P}_i(t_i(\lambda_+^*))$ with weight 1-a for each arm i.

Proof. For the optimal solution to (WHITTLE), recall that OPT denote the expected reward. The expected rate of playing the arms is exactly 1 by the LP constraint.

When $\lambda=0$, then $t_i(\lambda)=1$ for all i, implying $Q(\lambda)=n$. Similarly, when $\lambda=\lambda_{\max}\geq \max_i r_i$, $t_i(\lambda)=\infty$ for all i, so that $Q(\lambda)=0$. Therefore, as λ is increased from 0 to λ_{\max} , there is a transition value λ^* such that $Q(\lambda_-^*)=Q_1\geq 1$, and $Q(\lambda_+^*)=Q_2<1$; furthermore, $|\lambda_+^*-\lambda_-^*|\to 0$.

Since the solution to (WHITTLE) is feasible for LPLAGRANGE(λ), we must have:

$$R(\lambda_{+}^{*}) - \lambda^{*}Q_{2} \ge OPT - \lambda^{*}$$
 $R(\lambda_{-}^{*}) - \lambda^{*}Q_{1} \ge OPT - \lambda^{*}$

Let $a = \frac{1 - Q_2}{Q_1 - Q_2}$, then taking the convex combination of the above inequalities, we obtain:

$$aR(\lambda_{-}^{*}) + (1-a)R(\lambda_{+}^{*}) \ge OPT$$

$$aQ(\lambda_{-}^{*}) + (1-a)Q(\lambda_{+}^{*}) = 1$$

This completes the proof.

To prove Theorem 2.12, consider n i.i.d. arms with $n\beta \ll 1$, $\alpha = \beta/(n-1)$ and r=1. Each arm is in state g w.p. 1/n, so that all arms are in state b w.p. 1/e and the maximum possible reward of any feasible policy is 1-1/e even with complete information about the states of all arms.

We will show that Whittle's LP has value $1-O(\sqrt{n\beta})$ for $n\beta \ll 1$. Since the LP is symmetric w.r.t. the arms, it is easy to show (from Theorem A.1) that for each arm, it will construct the same convex combination of two single-arm policies. The first policy is of the form $\mathcal{P}(t-1)$, and the second is of the form $\mathcal{P}(t)$. The constraint is that if these policies are executed independently, exactly one arm is played in expectation per step. Since $\mathcal{P}(t)$ has lower average reward and rate of play than $\mathcal{P}(t-1)$, we consider the sub-optimal LP solution that uses policy $\mathcal{P}(t)$ for each arm.

The policy $\mathcal{P}(t)$ always plays in state g, and in state b, waits t steps before playing. The value t is chosen so that the rate of play for each arm is less than 1/n, and $\mathcal{P}(t-1)$ has a rate of play larger than 1/n. The rate of play of the single arm policy $\mathcal{P}(t)$ is given by the formula: $Q(t) = \frac{\beta + v_t}{t\beta + v_t}$. Since this is 1/n, we have $v_t = \beta(t-n)/(n-1)$. The reward of each arm is $R(t) = \frac{v_t}{t\beta + v_t} = \frac{t-n}{n(t-1)}$, so that the objective of Whittle's LP is $nR(t) = 1 - \Theta(n/t)$.

Now, from $v_t = \beta(t-n)/(n-1)$, we obtain $1-(1-\beta')^t = \beta'(t-n)$, where $\beta' = \alpha + \beta = \beta \frac{n}{n-1}$. This holds for $t = \Theta(\sqrt{n/\beta})$ provided $n\beta \ll 1$. Plugging this value of t into the value nR(t) of Whittle's LP completes the proof of Theorem 2.12.

A.4 Proof of Lemma 3.1

Recall the notation from Section 2.2 and Definition 3. We first present the following structural lemma about the optimal single-arm policy $L_i(\lambda)$. Suppose this policy is of the form $\mathcal{P}_i(t_i(\lambda))$, where $t_i(\lambda) = \operatorname{argmax}_{t>1} F_i(\lambda,t)$.

Lemma A.2. $t_i(\lambda)$ is monotonically non-decreasing in λ .

Proof. We have: $\frac{\partial F_i(\lambda,t)}{\partial \lambda} = -Q_i(t) = -\frac{v_{it}+\beta_i}{v_{it}+t\beta_i}$. Since $Q_i(t)$ is a decreasing function of t, the above is an increasing function and always negative, which implies that for smaller t, the function $F_i(\lambda,t)$ decreases faster as λ is increased. This implies that if $t_i(\lambda) = \operatorname{argmax}_{t\geq 1} F_i(\lambda,t)$, then for $\lambda' \geq \lambda$, the maximum of $F_i(\lambda',t)$ is attained for some $t_i(\lambda') \geq t_i(\lambda)$.

Now note that when $\lambda = 0$, there is no penalty, so that the single-arm policy maximizes its reward by playing every step regardless of the state. Therefore, $\Pi_i(s,t) \geq 0$ for all states $(s,t)^3$.

Suppose the arm is in state (g,1). The immediate expected reward if played is $r_i(1-\beta_i)$. If the penalty $\lambda < r_i(1-\beta_i)$, a policy that plays the arm and stops later has positive expected reward minus penalty. Therefore, for penalty λ , the optimal decision at state (g,1) is "play", so that $\Pi_i(g,1) \geq r_i(1-\beta_i)$. We now show that $\Pi_i(g,1) = r_i(1-\beta_i)$. Suppose the penalty is $\lambda > r_i(1-\beta_i)$. If played in state (g,1), the immediate expected reward minus penalty is negative, and leads to the policy being in state (g,1) or (b,1). The best possible total reward minus penalty in the future is obtained by always playing in state (g,1) and waiting as long as possible in state (b,1) (since this maximizes the chance of going to state g if played). Whenever the arm is played in state g after g steps, the probability of observing state g is at most $\frac{\alpha_i}{\alpha_i+\beta_i}$. Consider two consecutive events of the policy when the last play was in state (g,1) and the current observed state is (g,1). Since the optimal such policy is ergodic, this interval would define a renewal period. In this period, the expected penalty is at least $\lambda\left(\frac{\alpha+\beta}{\alpha}+\frac{1}{\beta}\right)$, and the expected reward is $\frac{r_i}{\beta_i}$. Therefore, the next expected reward minus penalty in the renewal period is:

$$\frac{r_i - \lambda}{\beta_i} - \lambda \frac{\alpha_i + \beta_i}{\alpha_i} < r_i \left(1 - (1 - \beta_i) \left(1 + \frac{\beta_i}{\alpha_i} \right) \right) = -r_i \frac{\beta_i}{\alpha_i} (1 - \beta_i - \alpha_i) < 0$$

The last inequality follows since $\alpha_i + \beta_i \leq 1 - \delta$ for a $\delta > 0$ specified as part of input. This implies that if $\lambda > r_i(1 - \beta_i)$, the any policy that plays in state (g, 1) has negative net reward minus penalty, showing that "not playing" is optimal. Therefore, $\Pi_i(g, 1) = r_i(1 - \beta_i)$.

Next assume that for penalty slightly less than $r_i(1-\beta_i)$, the policy decision is to "play" in state (b,t). Consider the smallest such t. Since the policy also decides to play in (g,1), consider the renewal period defined by two consecutive events where the policy when the last play was in state (g,1) and the current

³Note that this is true *only* for FEEDBACK MAB where the underlying 2-state process evolves regardless of the plays; the claim need not be true for MONOTONE bandits defined in Section 4, where even with penalty $\lambda = 0$, the arm may idle in certain states.

observed state is (b,1). The reward is $\frac{r_i}{\beta_i}$ and the penalty is $\lambda\left(\frac{1}{v_{it}}+\frac{1}{\beta_i}\right)$. Since $\lambda=r_i(1-\beta_i)$, and $v_{it}<\frac{\alpha_i}{\alpha_i+\beta_i}$, the above analysis shows that the net expected reward minus penalty is negative in renewal period. Therefore, the decision in (b,t) is to "not play", so that $\Pi_i(b,t)\leq r_i(1-\beta_i)$.

Finally, for any $\lambda < r_i(1-\beta_i)$, consider the smallest $t \geq 1$ so that the optimal decision in state (b,t) is to "play". If this is finite, the optimal policy for this λ is precisely $L_i(\lambda) = \mathcal{P}_i(t_i(\lambda))$. From Lemma A.2, the function $t_i(\lambda)$ is non-decreasing in λ . Therefore, for any state (b,t^*) , the quantity $\max\{\lambda|L_i(\lambda)=\mathcal{P}_i(t^*)\}$ is well-defined. For larger values of penalty λ , we have $t_i(\lambda) > t^*$, so that the decision in (b,t^*) is "do not play". Therefore, $\Pi_i(b,t) = \max\{\lambda|L_i(\lambda)=\mathcal{P}_i(t)\}$. Since $t_i(\lambda)$ is non-decreasing in λ , the function $\Pi_i(b,t)$ is non-decreasing in t. This completes the proof of Lemma 3.1.